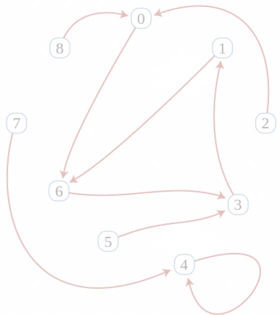
Element Distinctness, Birthday Paradox, and 1-out Pseudorandom Graphs

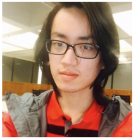


Hongxun Wu

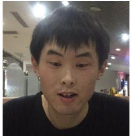
IIIS, Tsinghua University

42	3	23	1	12	30	42	15

Authors of this work



Lijie Chen



Ce Jin

R. Ryan Williams

我

Hongxun Wu

Lijie Chen, Ce Jin, and R. Ryan Williams are from MIT.

Element Distinctness

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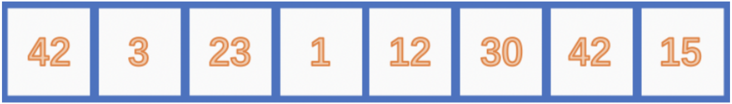

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- INPUT: n positive integers $a_1, a_2, \ldots, a_n$ with $a_i \leq \text{poly}(n)$.
- Decide whether all a's are distinct.

Element Distinctness

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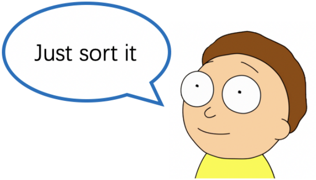

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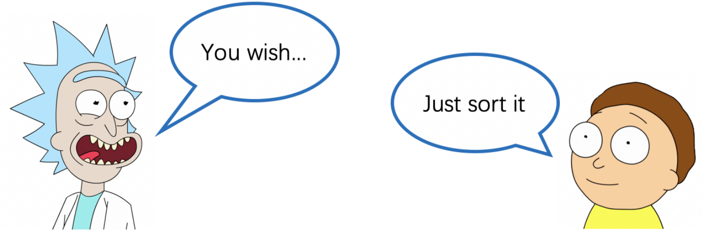
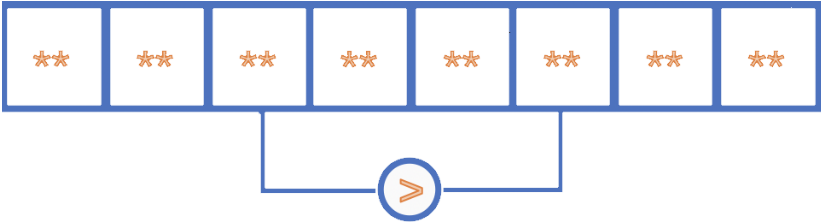
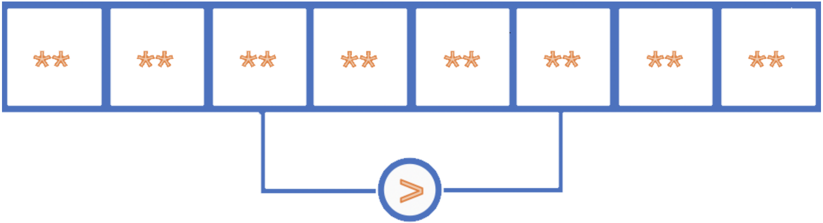


Comparision model



- No direct access to the INPUT a.
- Each query (i, j) returns one of $a_i < a_j$, $a_i = a_j$, $a_i > a_j$.

Comparision model



Time-Space tradeoff [BFMADH$^+$87, Yao88]

Element distictness requires $TS = \Omega\left(n^{2-o(1)}\right)$ in Comparision model.

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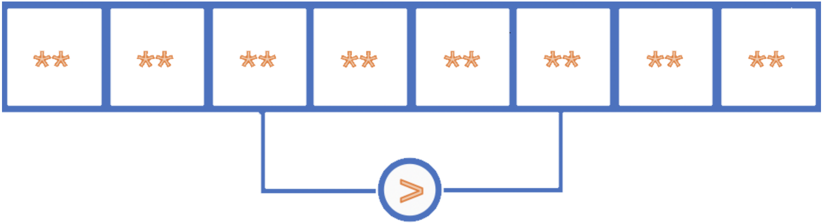

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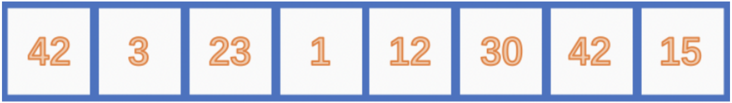
Element distictness requires $TS = \Omega\left(n^{2-o(1)}\right)$ in Comparision model.

- When $S = O(\text{polylog}\, n)$, $T = \Omega\left(n^{2-o(1)}\right)$.

RAM model



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- Random access to read-only input.
- Working memory has a (relatively small) size S.

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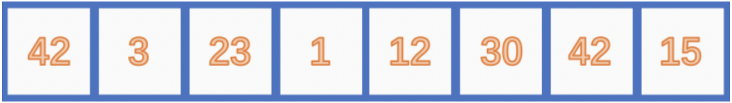

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Time-Space tradeoff [BCM13]

- Assuming the existence of *Random Oracle*, there is an algorithm with $T^2S = \tilde{O}\left(n^3\right)$.

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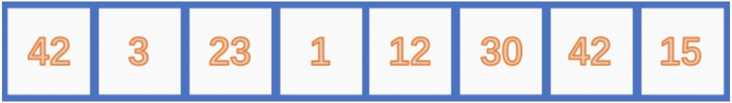

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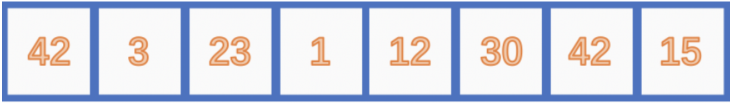

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- When $S = \tilde{O}(1)$, $T = \tilde{O}\left(n^{1.5}\right)$.
- In the rest of this talk, we always assume there is only one collision $(a_i = a_j)$.

1-out Graph and Birthday Paradox

Pollard's ρ method [BCM13]

Assuming the existence of *Random Oracle*, when $S = \tilde{O}(1)$, there is an algorithm with $T = \tilde{O}\left(n^{1.5}\right)$.

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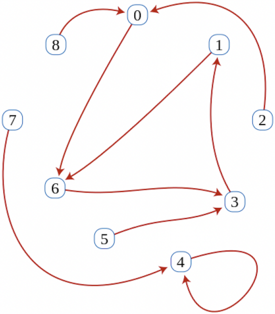

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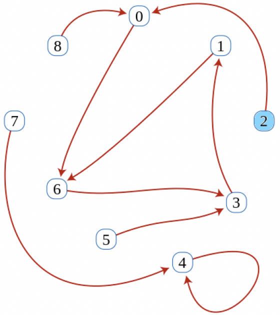

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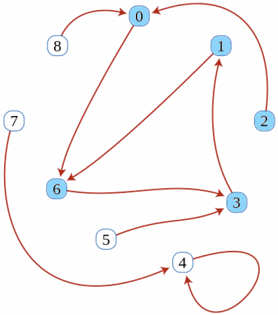
- Pick a random starting point s.
- Run Floyd's cycle finding.

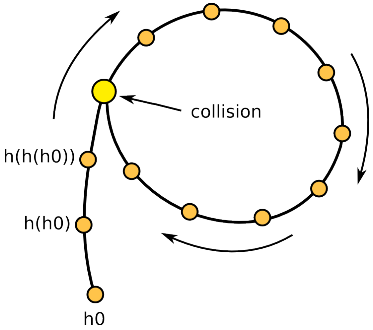


1-out Graph and Birthday Paradox

Birthday Paradox Type Properties [BCM13]

Suppose $f^*(s)$ is the set of vertices reachable from s.

- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$



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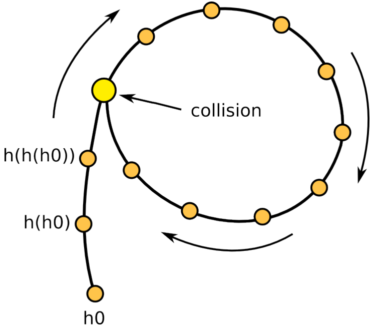

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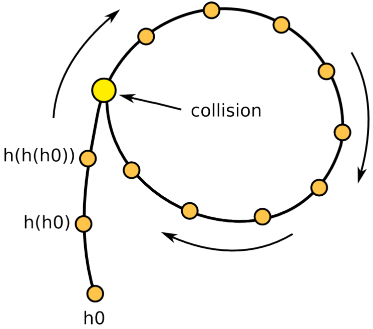

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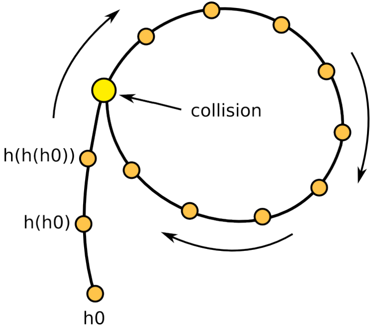
- Repeat $O(n)$ times. It takes $O(n^{1.5})$ time in total.



Our Results

Our Main Lemma

There exsits a family $\{r_{\text{seed}}\}$ of hash functions efficiently samplable with seed length $O(\operatorname{polylog} n)$, and the graph defined by $\{r_{\text{seed}}\}$ (instead of *Random Oracle R*) satisfy

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~~Assuming the existence of *Random Oracle*~~, when $S = O(\operatorname{polylog} n)$, there is a RAM algorithm for Element Distinctness with $T = \tilde{O}\left(n^{1.5}\right)$.

Subset Sum

Low-space Algorithm for Subset Sum [BGNV18]

Assuming the existence of *Random Oracle*, Subset Sum and Knapsack can be solved by a Monte Carlo algorithm in $O^*(2^{0.86n})$ time, with $O(\text{poly}(n))$ space.

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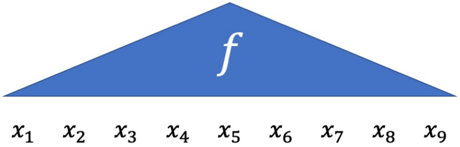
Construction

Random Restriction and Håstad's Switching Lemma

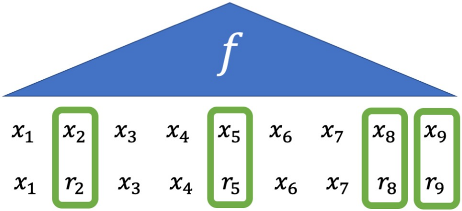
This is Ryan O'Donnell's Youtube lecture which is a masterpiece.

Iterative Restriction



This is the Ajtai-Wigderson Paradigm [AW85] for building PRG.

Iterative Restriction



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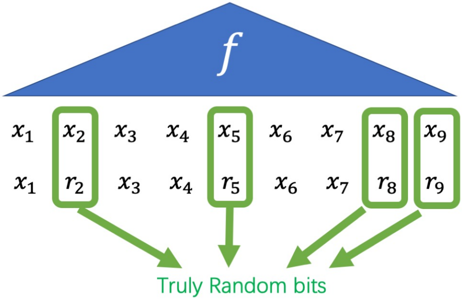

f
x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9
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Truly Random bits

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Toy Example: Two levels

Recall the input $a_1, a_2, \ldots, a_n \in [m]$.

Two Level Example

Suppose we have the following:

- $O(\text{polylog } n)$-wise independent functions $g : [m] \to \{0, 1\}$ and $r : [m] \to [n]$.
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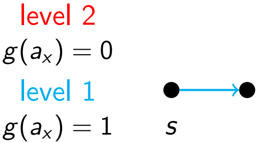

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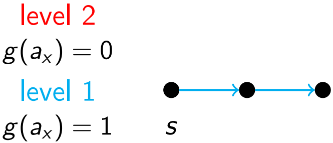

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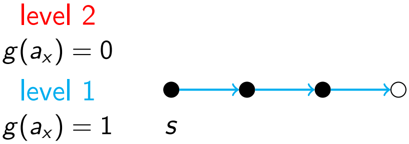

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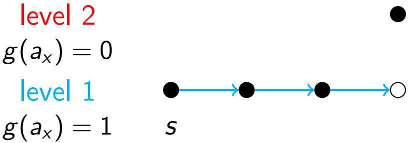

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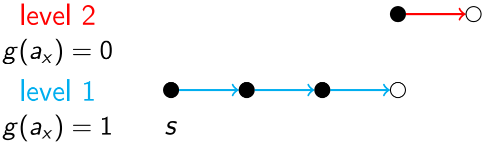

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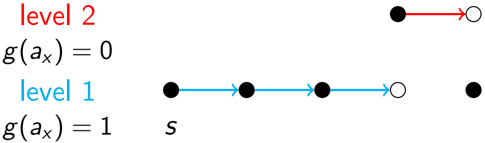

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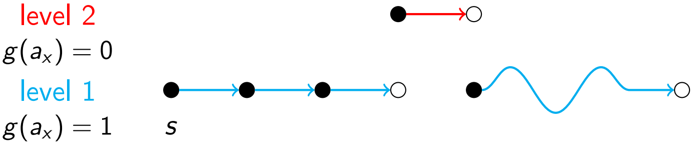

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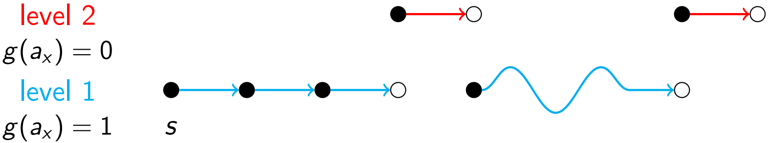

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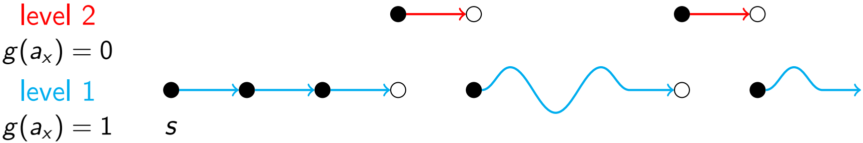

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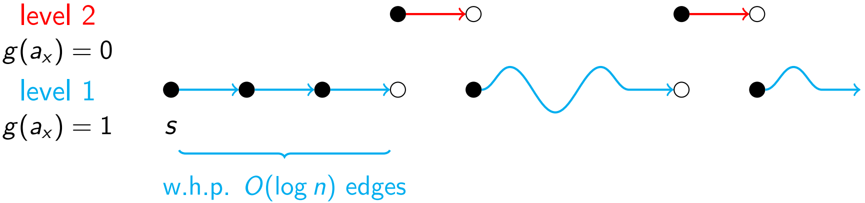
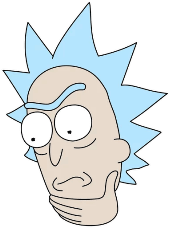


Sanity Check

- Why this might be a good idea?



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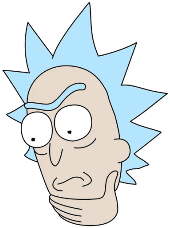

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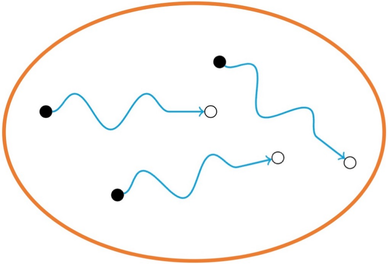

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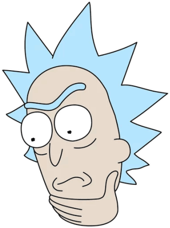

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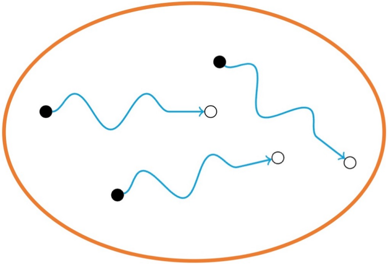

- Each subpath has length $O(\log n)$.
- Every level 2 edge is an independent sample of a subpath.

Intuition: Memory Eraser

Recall our goal.

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- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$
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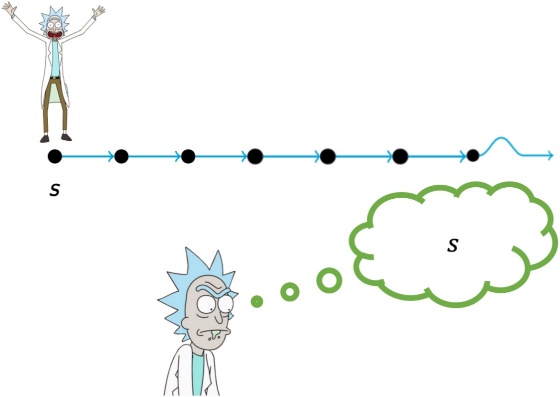

- “Memory” of a random walk: The current vertex it is at.

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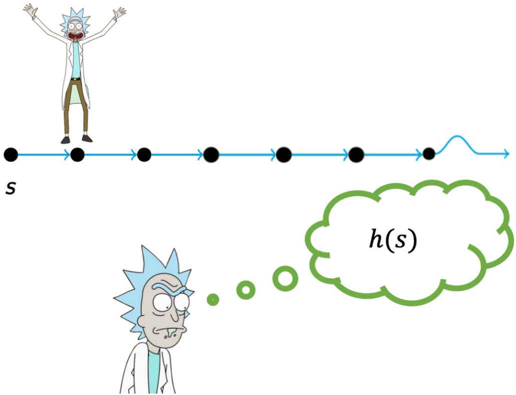

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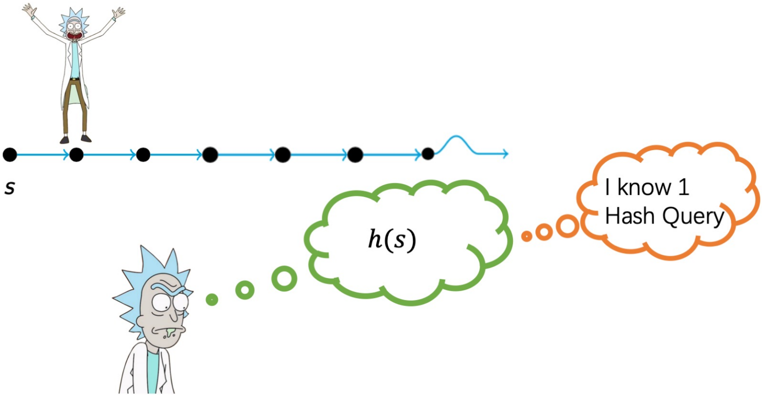
Intuition: Memory Eraser



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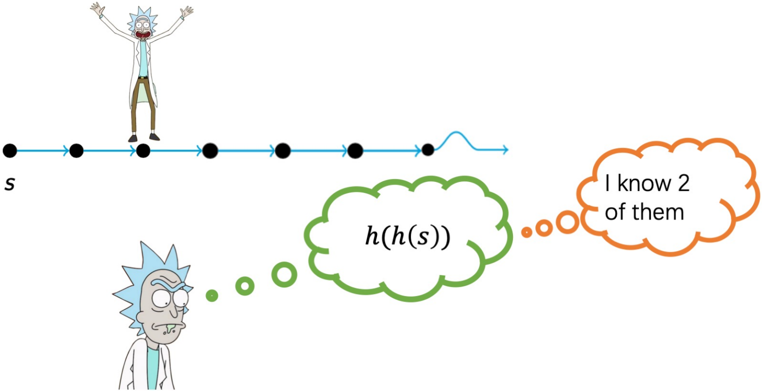

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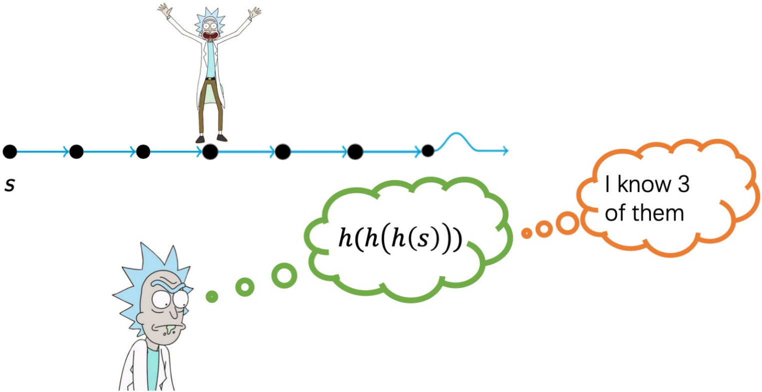

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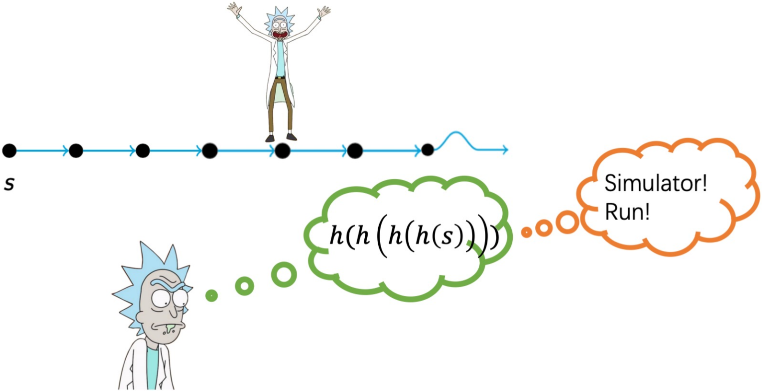

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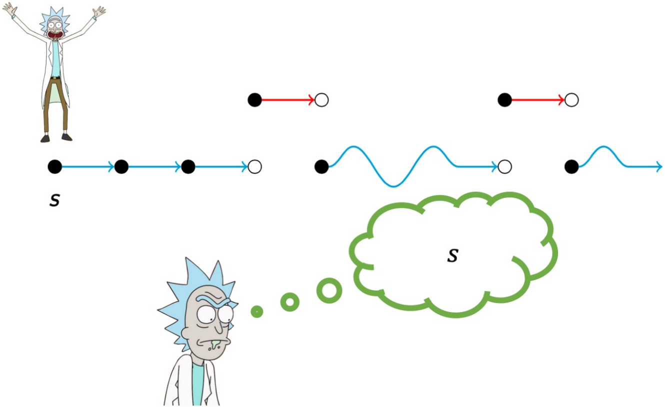

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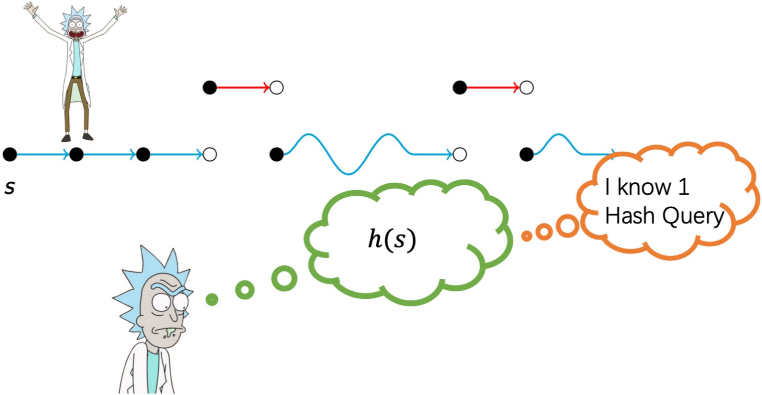

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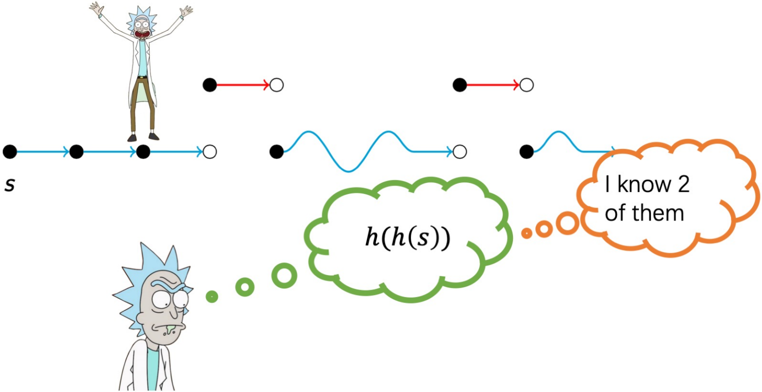

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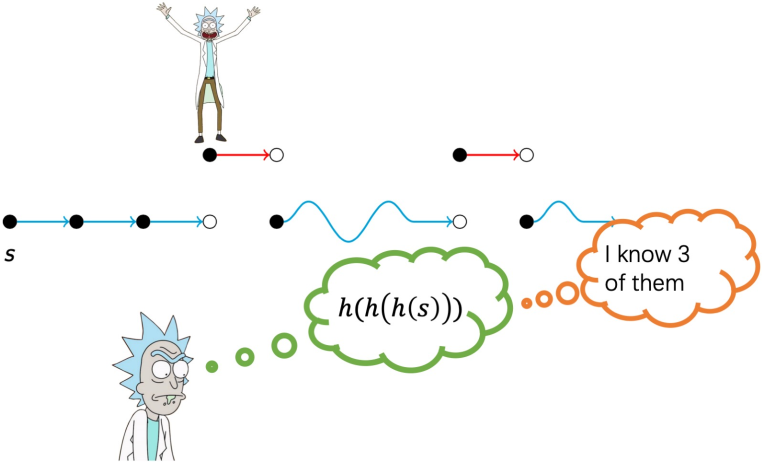

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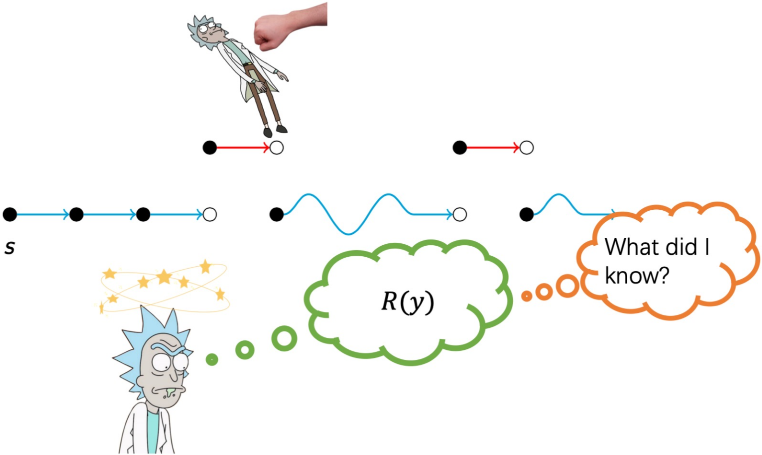

- "Memory" of a random walk: The current vertex it is at.

Our Construction via Iterative Restriction

Our Construction

Now we sample $O(\log n)$ many hash functions $\{r_i, g_i\}_{i \in [\ell]}$.
Each $r_i : [m] \to [n]$ and $g_i : [m] \to [2]$ are $O(\log n)$-wise independent.

Our Construction via Iterative Restriction

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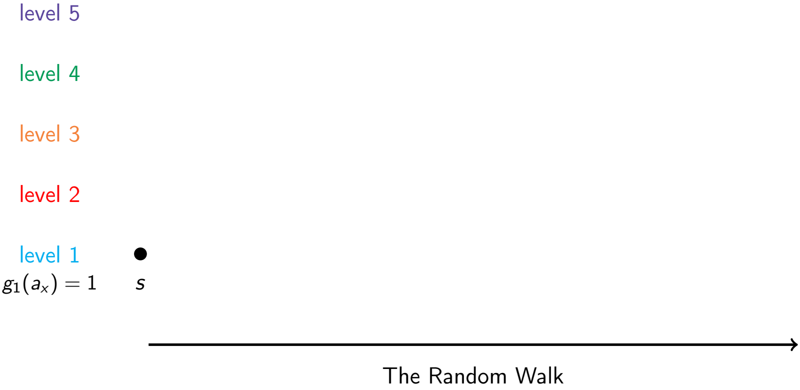
Each $r_i : [m] \to [n]$ and $g_i : [m] \to [2]$ are $O(\log n)$-wise independent.
Then we set $h_{\ell+1}(a_x) = \perp$ and

$$h_i(a_x) = \begin{cases} h_{i+1}(a_x) & g_i(a_x) = 0 \\ r_i(a_x) & g_i(a_x) = 1 \end{cases}$$

Finally, we set $h = h_1$.

Our Construction via Iterative Restriction



level 5

level 4

level 3

level 2

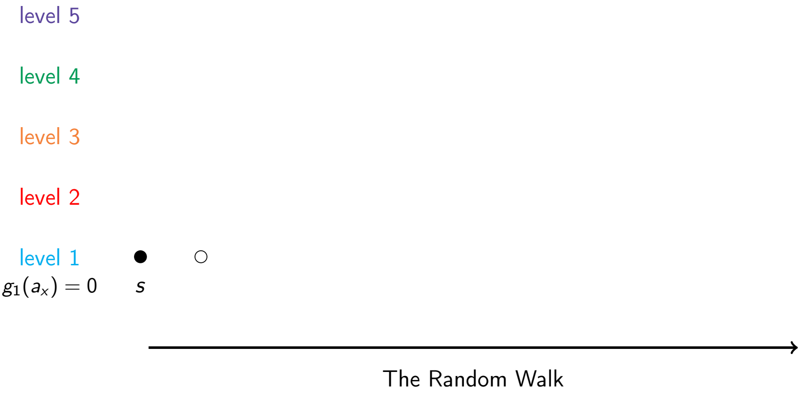
level 1

$g_1(a_x) = 1$

s

The Random Walk

Our Construction via Iterative Restriction



level 5

level 4

level 3

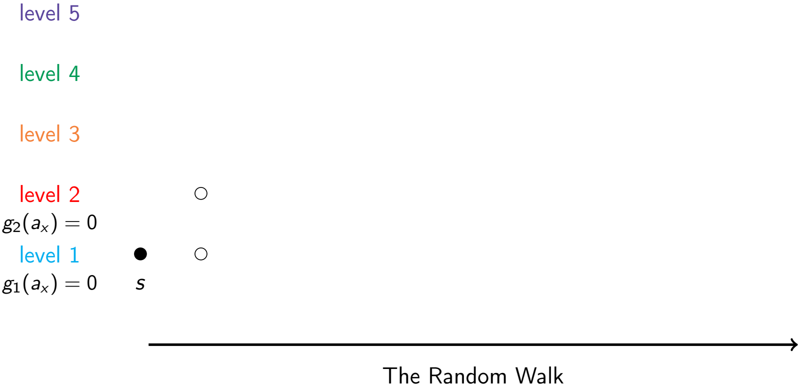
level 2

level 1

$g_1(a_x) = 0$ s

The Random Walk

Our Construction via Iterative Restriction



level 5

level 4

level 3

level 2

$g_2(a_x) = 0$

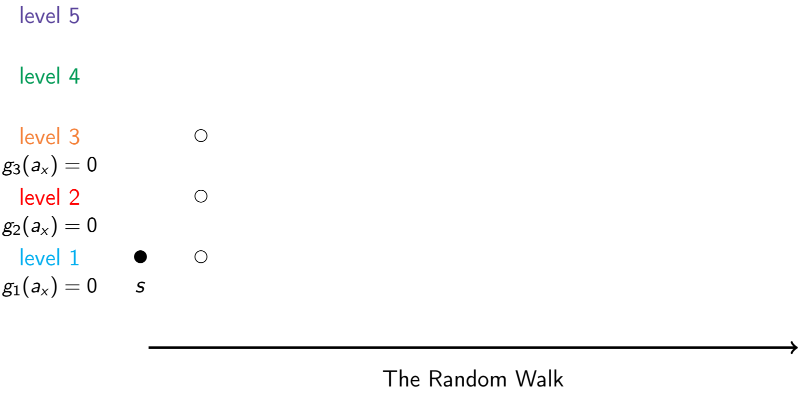
level 1

$g_1(a_x) = 0$

s

The Random Walk

Our Construction via Iterative Restriction



level 5

level 4

level 3

$g_3(a_x) = 0$

level 2

$g_2(a_x) = 0$

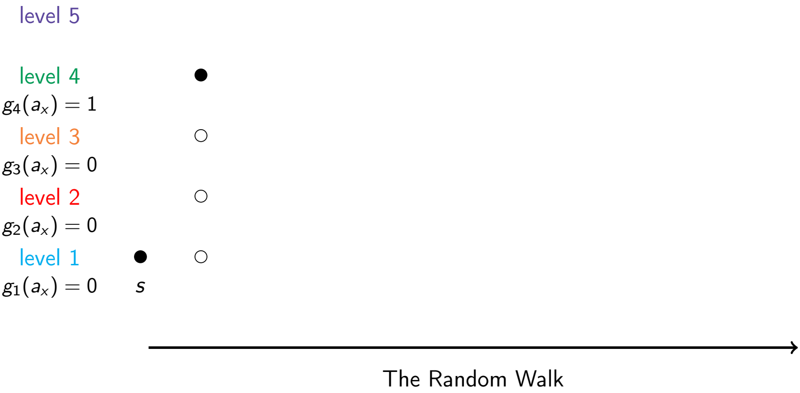
level 1

$g_1(a_x) = 0$

s

The Random Walk

Our Construction via Iterative Restriction



level 5

level 4

$g_4(a_x) = 1$

level 3

$g_3(a_x) = 0$

level 2

$g_2(a_x) = 0$

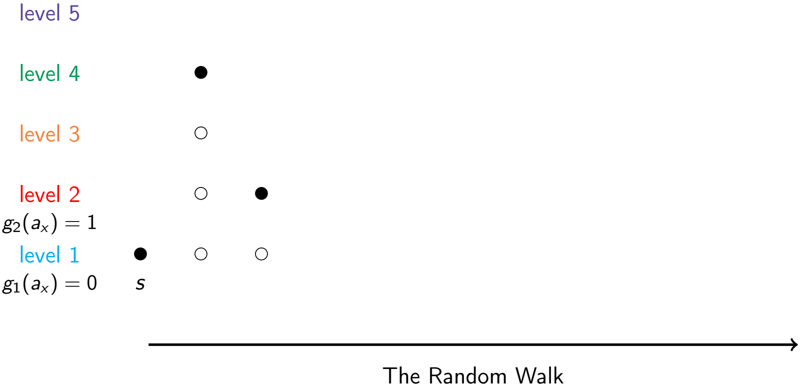
level 1

$g_1(a_x) = 0$

s

The Random Walk

Our Construction via Iterative Restriction



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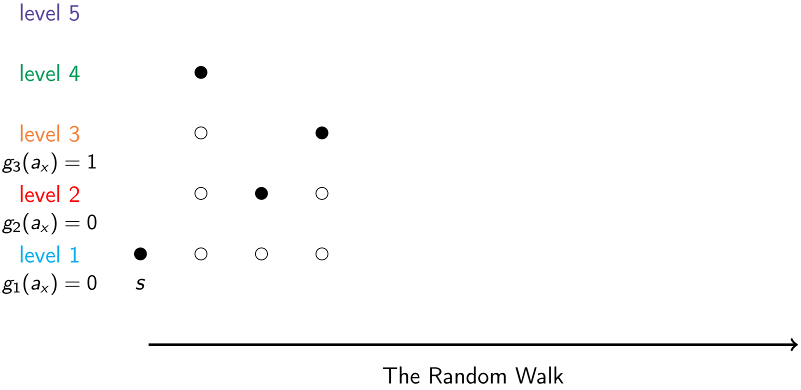

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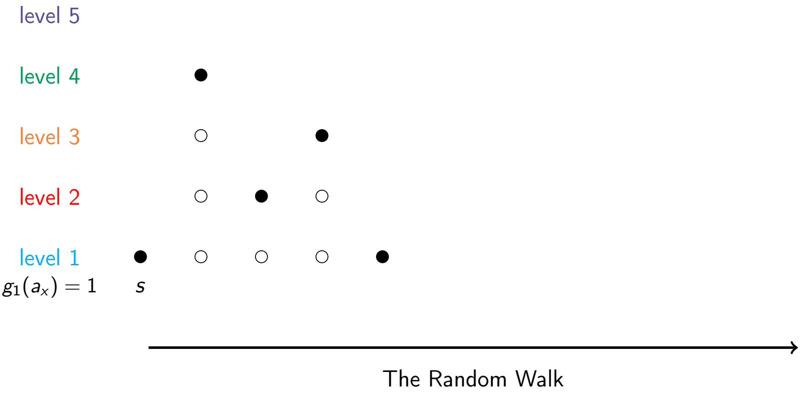

level 5

level 4

level 3

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level 1

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The Random Walk

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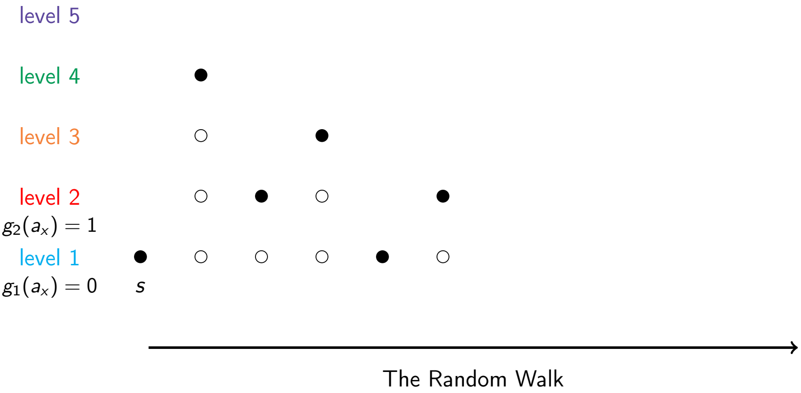

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s

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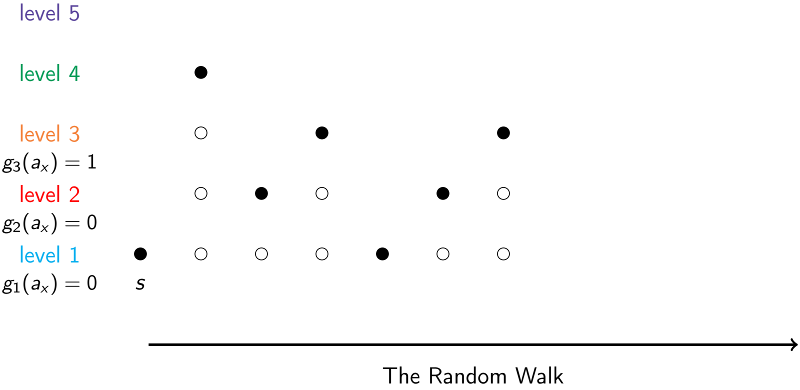

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level 4

level 3

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level 2

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level 1

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s

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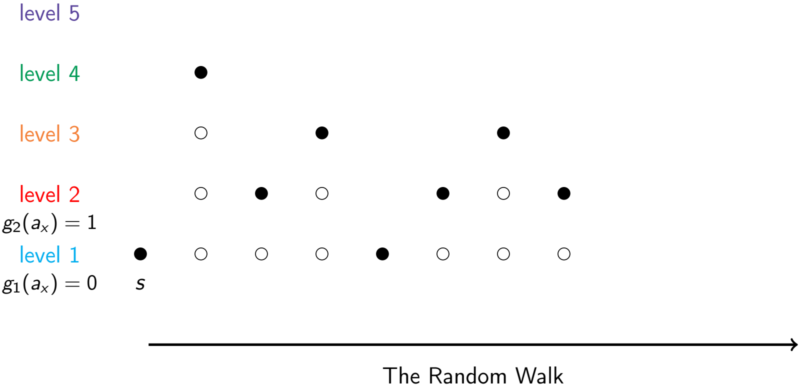

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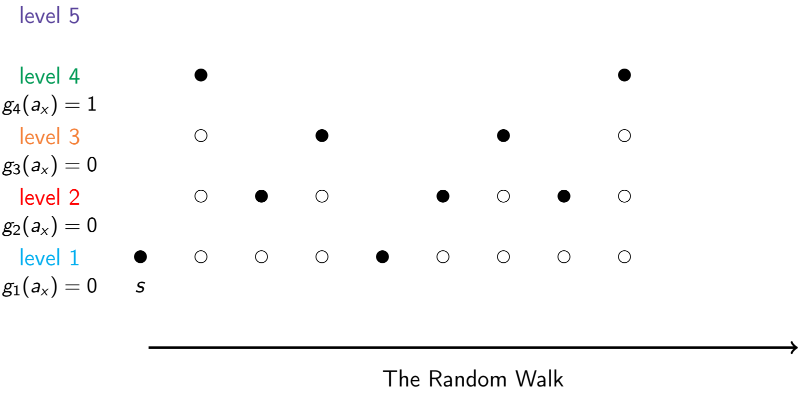

level 5

level 4

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level 3

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level 1

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s

The Random Walk

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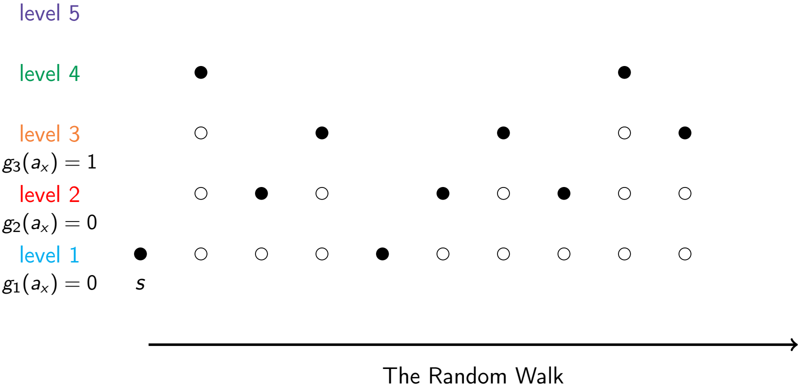

level 5

level 4

level 3

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level 2

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s

The Random Walk

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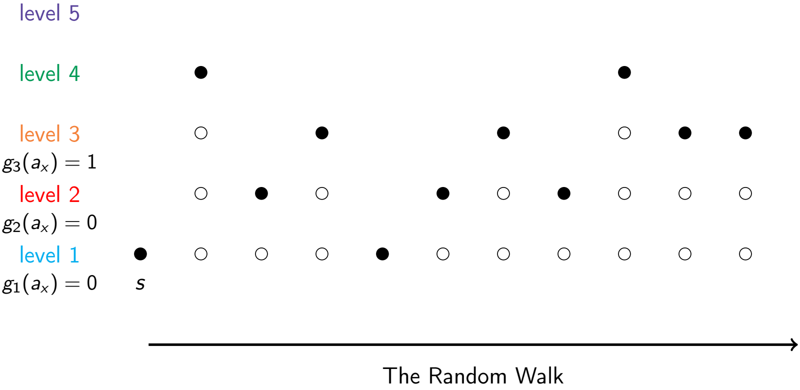

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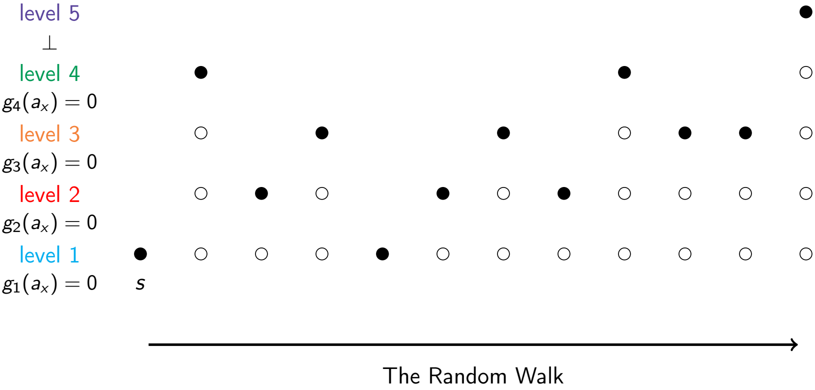
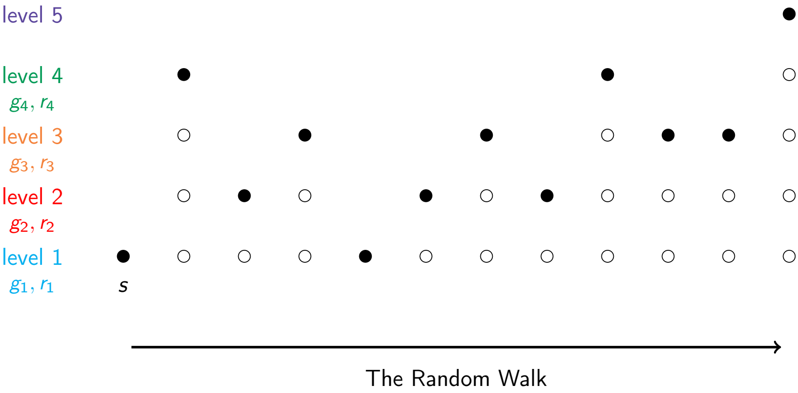
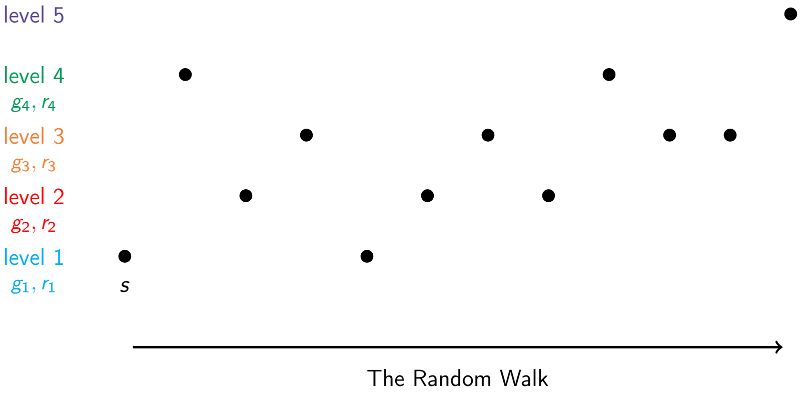


Key Ideas in Our Analysis

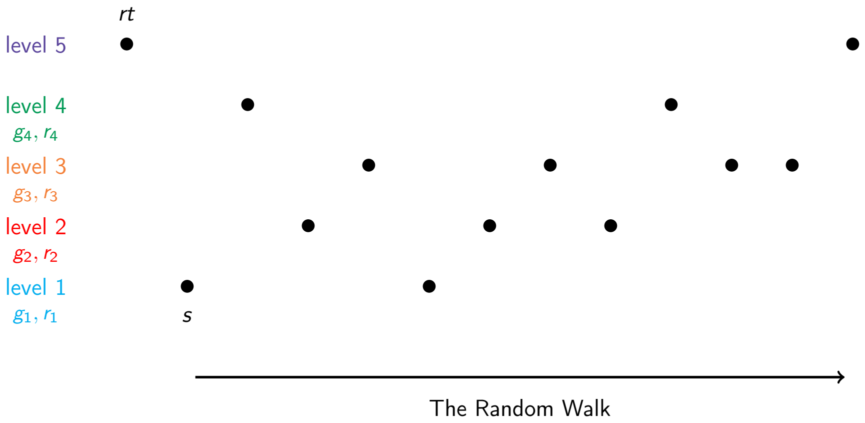
Dependency Tree



Dependency Tree



Dependency Tree



rt

level 5

level 4

g_4, r_4

level 3

g_3, r_3

level 2

g_2, r_2

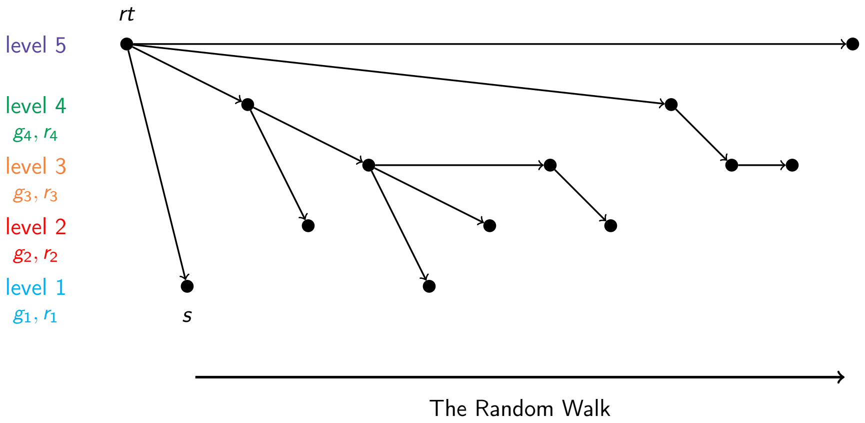
level 1

g_1, r_1

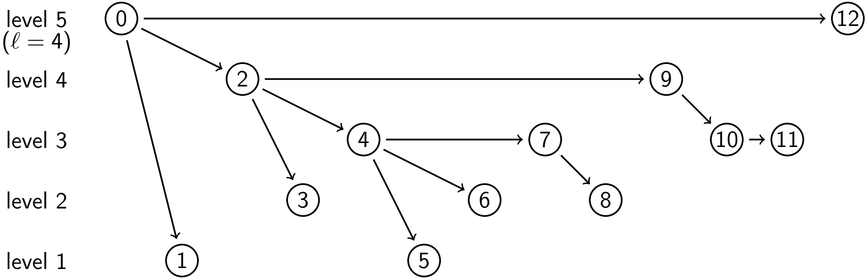
s

The Random Walk

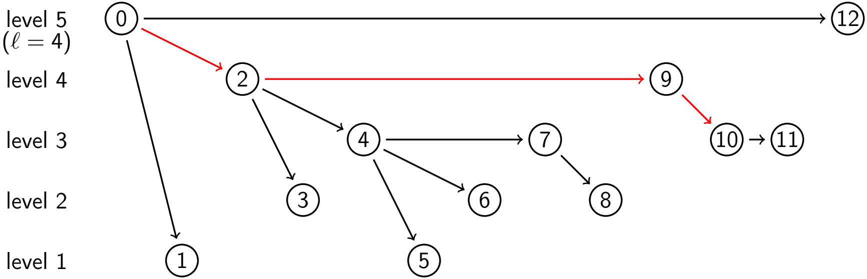
Dependency Tree



Dependency Tree

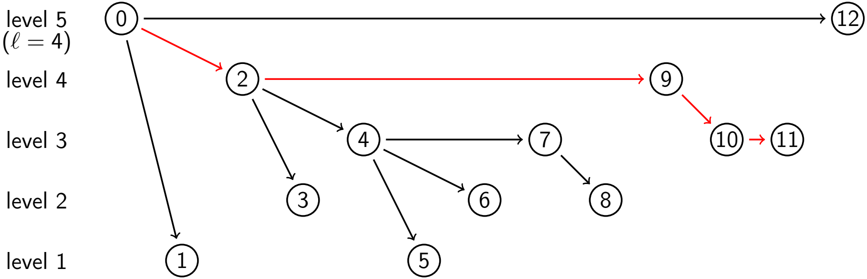


Dependency Tree



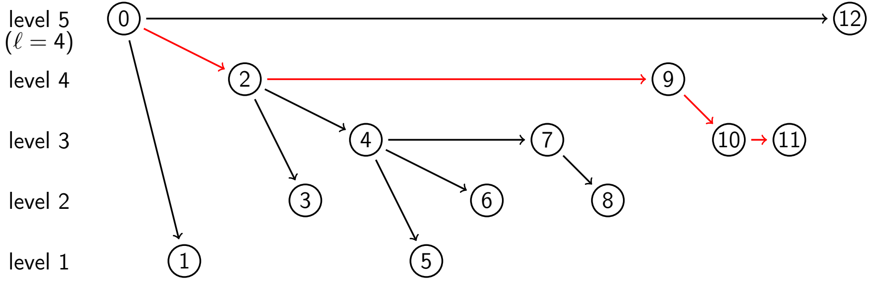
- We index a node by the shape of its path, e.g. $\vec{k}_{10} = (0, 0, 1, 2)$.

Dependency Tree



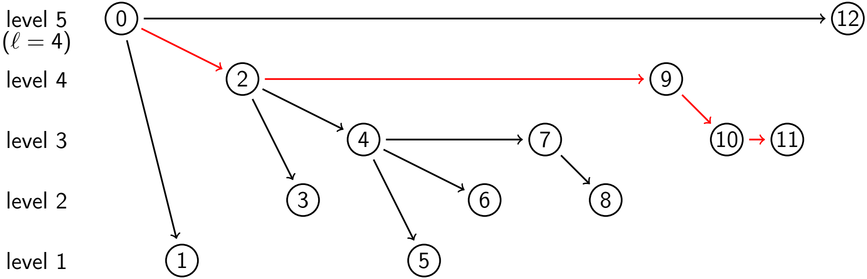
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.

Dependency Tree



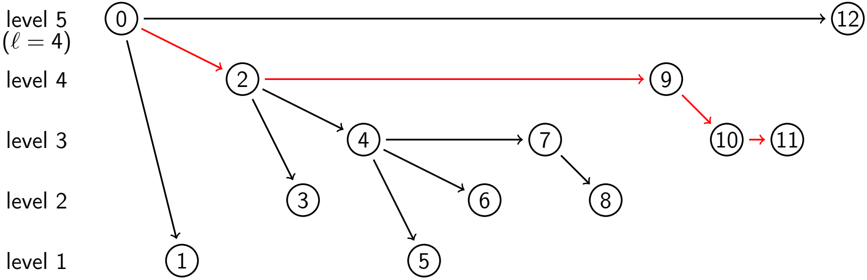
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider $\vec{k}_x$. Fix x, $\vec{k}$ is a random variable. Fix $\vec{k}$, x is a random variable.

Dependency Tree



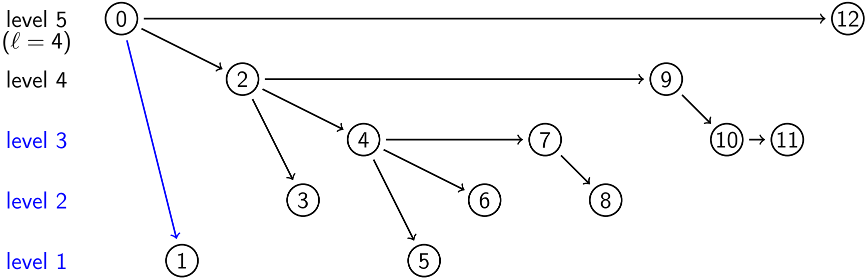
- We index a node by the shape of its path, e.g. $\vec{k}_{11} = (0, 0, 2, 2)$.
- Consider $\vec{k}_x$. Fix x, $\vec{k}$ is a random variable. Fix $\vec{k}$, x is a random variable.
- We fix index $\vec{k}$ and let x be the random variable (which may not exist).

Memory Eraser on Dependency Tree



- Fix $\vec{k} = (0, 0, 2, 2)$.

Memory Eraser on Dependency Tree



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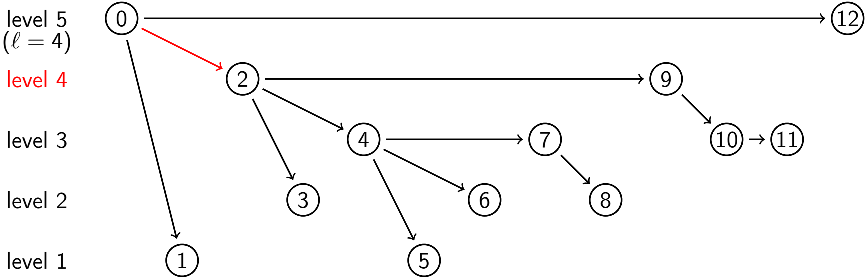

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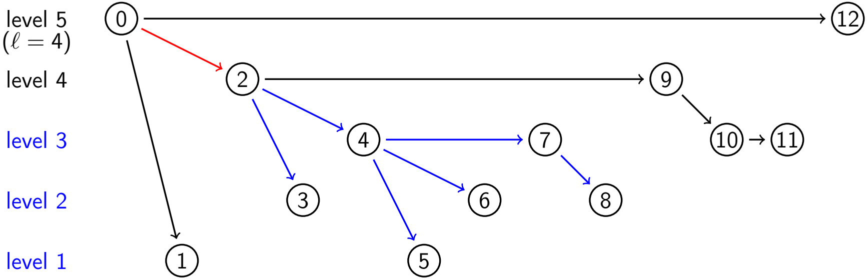
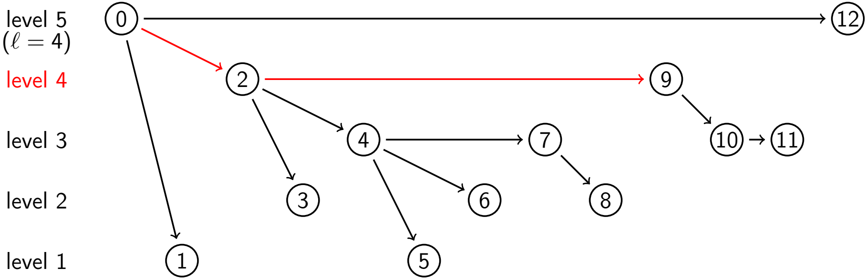


- Fix $\vec{k} = (0, 0, 2, 2)$.
- Blue part is a random variable. But it will finally end up with a node with level ≥ 4.

Memory Eraser on Dependency Tree



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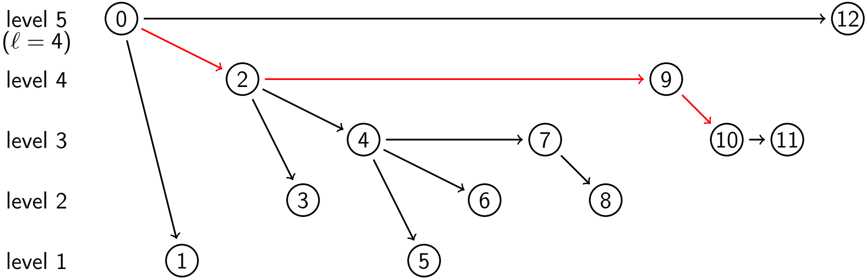

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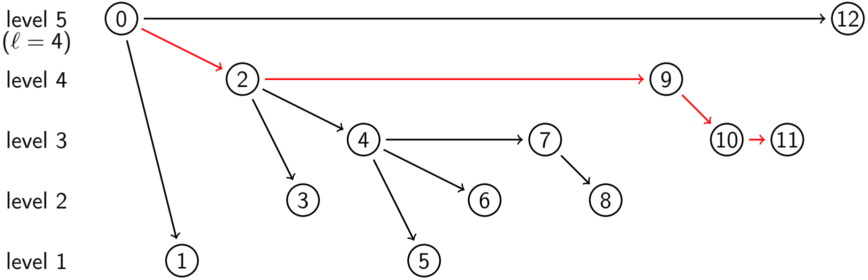

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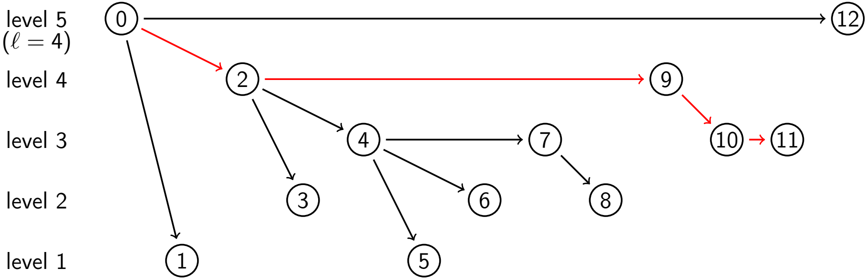
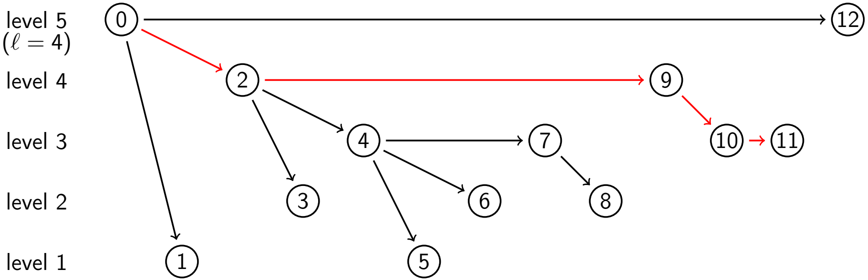
Memory Eraser on Dependency Tree



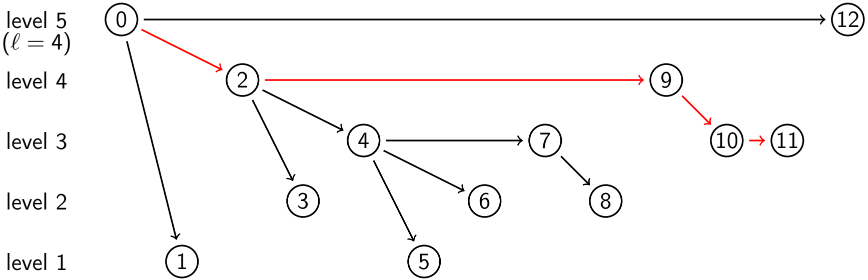
- Fix $\vec{k} = (0, 0, 2, 2)$.
- Blue part is a random variable. But it will finally end up with a node with level ≥ 4.
- One issue: What if $a_{w_2} = a_{w_9}$?

(Locally Simulatable) Extended Walk



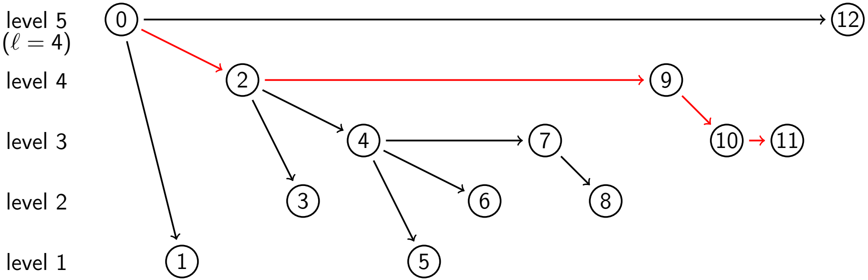
- Instead of original walk w, we look at extended walk w^*.

(Locally Simulatable) Extended Walk



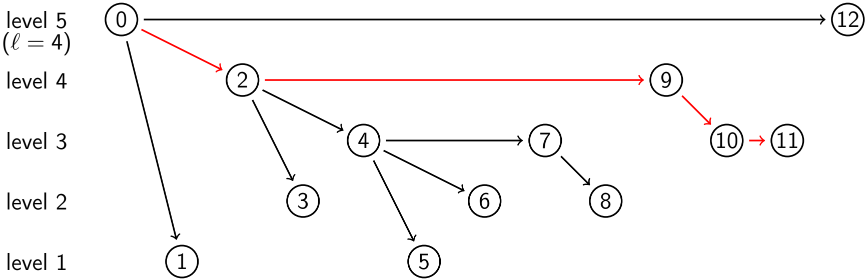
- Instead of original walk w, we look at extended walk w^*.
- Once a position in our memory is queried twice, we replace it with true randomness.

(Locally Simulatable) Extended Walk



- Instead of original walk w, we look at extended walk w^*.
- Once a position in our memory is queried twice, we replace it with true randomness.
- w^* is *locally simulatable* in the sense that each query position can be uniquely determined by memory.

(Locally Simulatable) Extended Walk



- Instead of original walk w, we look at extended walk w^*.
- Once a position in our memory is queried twice, we replace it with true randomness.
- w^* is *locally simulatable* in the sense that each query position can be uniquely determined by memory.
- w and w^* agree if w^* has no collision $a_{w_i^*} = a_{w_j^*}$.

Good = All - Bad

Recall our goal.

Our Main Lemma

- $\mathbb{E}[|f^*(s)|] \leq O(\sqrt{n})$
- $\Pr[u \in f^*(s)] \geq \Omega(1/\sqrt{n}), \ \forall u \in [n]$

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- Good: $E[\#\{t|w_t^* = u, w^* \text{ has no collision}\}] \geq \Pr[\exists t, w_t = u]$.

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- All: $E[\#\{t | w_t^* = u\}]$

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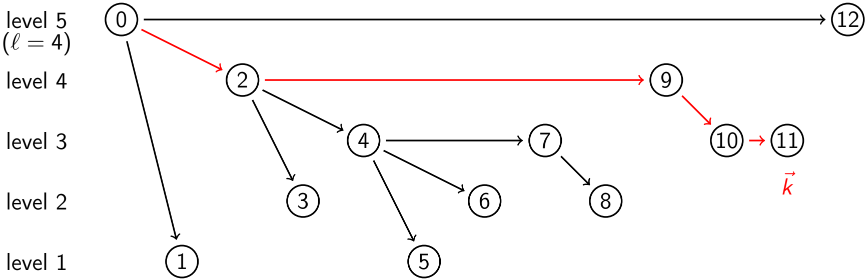
- All: $E[\#\{t|w_t^* = u\}]$
- Bad:

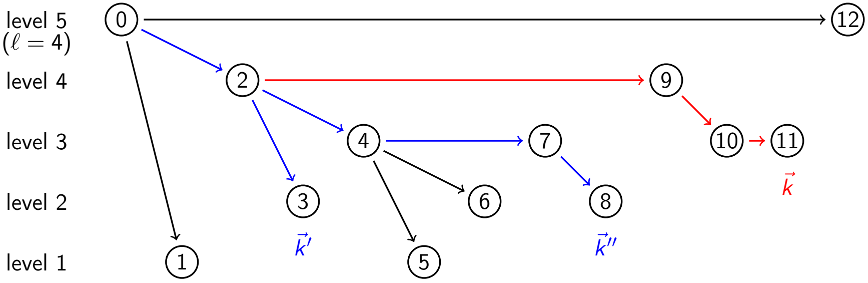
$$E[\#\{t|w_t^* = u, \exists t' \neq t'', a_{w_{t'}^*} = a_{w_{t''}^*}\}]$$
$$\leq E[\#\{t, t' \neq t''|w_t^* = u, a_{w_{t'}^*} = a_{w_{t''}^*}\}]$$

Good = All - Bad



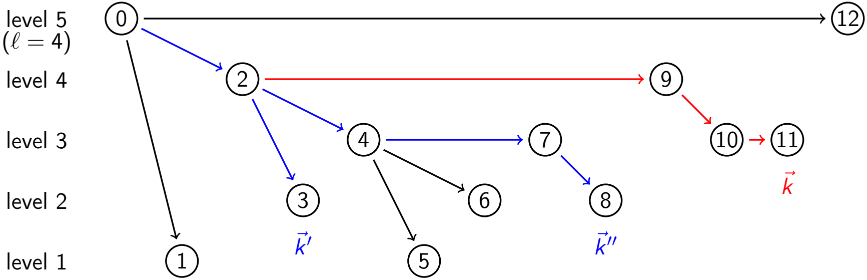
$$E[\#\{t | w_t^* = u\}] = \sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n}$$

Good = All - Bad



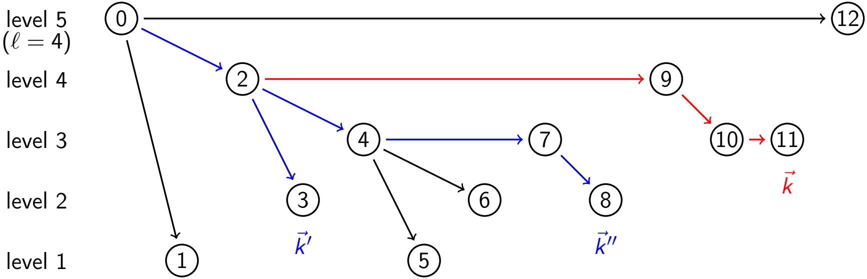
$$E[\#\{t, t' \neq t'' | w_t^* = u, a_{w_{t'}^*} = a_{w_{t''}^*}\}] = \sum_{\vec{k}, \vec{k}', \vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^2}$$

Good = All - Bad



$$\text{Good} = \sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1 - \|\vec{k}'\|_1 - \|\vec{k}''\|_1}}{n^2}$$

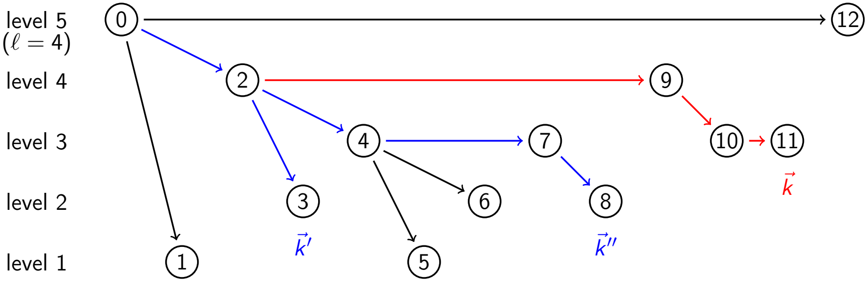
Good = All - Bad



$$\text{Good} = \sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^2}$$

$$\sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} = \frac{1}{n}\prod_{i=1}^{\ell}\sum_{k_i=0}^{\infty} 2^{-k_i} = \frac{2^\ell}{n}$$

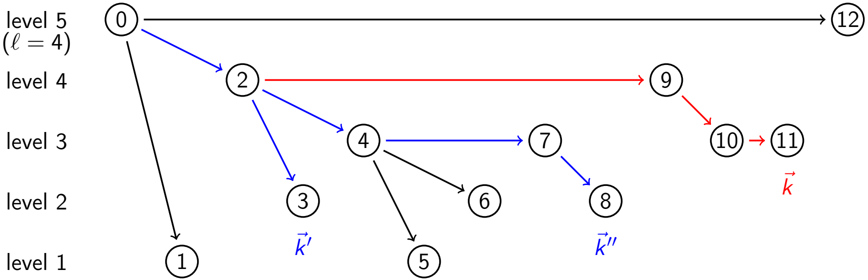
Good = All - Bad



$$\text{Good} = \sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^2}$$

$$\sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^2} = \frac{8^\ell}{n^2}$$

Good = All - Bad



$$\text{Good} = \sum_{\vec{k}} \frac{2^{-(k_1+k_2+\cdots+k_\ell)}}{n} - \sum_{\vec{k},\vec{k}',\vec{k}''} \frac{2^{-\|\vec{k}\|_1-\|\vec{k}'\|_1-\|\vec{k}''\|_1}}{n^2} = \frac{2^\ell}{n} - \frac{8^\ell}{n^2}$$

Let $\ell \leftarrow \frac{1}{2}\log n - 100$. $\frac{2^\ell}{n} - \frac{8^\ell}{n^2} = \frac{2^{-100}}{\sqrt{n}} - \frac{2^{-300}}{\sqrt{n}} = \Omega\left(\frac{1}{\sqrt{n}}\right)$.

Warning

Our Main Lemma

- $\mathbb{E}[|f^*(s)|] \le O(\sqrt{n})$
- $\Pr[u \in f^*(s)] \ge \Omega(1/\sqrt{n}),\ \forall u \in [n]$

- Even for this simple case, there is so much more technical challenges that is hidden in this talk.

Open Problems

Open Problems

- **Time-space Tradeoffs**
  In this work, we only solved the case when $S = \tilde{O}(1)$. Can we extend it to the full tradeoff?
- **Shorter Seed Length**
  In this work, our seed length is $O(\log^3 n \log\log n)$. Can this be improved?

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