Good Contention Resolution Schemes Cannot Be Oblivious for Matroids

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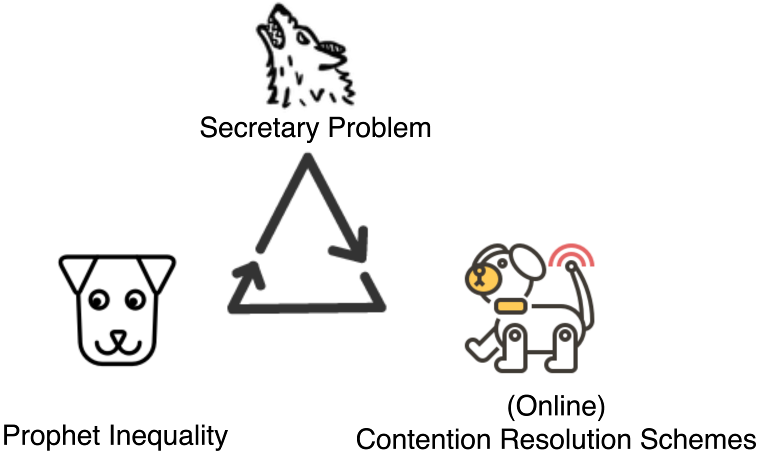
Background


Secretary Problem
Prophet Inequality
(Online)
Contention Resolution Schemes

Secretary Problem

v_0 v_1 v_2 v_3

- Each candidate has a value v_i.
- They come in a **random** order.
- You must irrevocably decide which candidate to hire.
- Maximize $\mathbb{E}[v_{\text{hire}}]$ / the probability to hire the best one.
- $\mathbb{E}[v_{\text{hire}}] \geq \frac{1}{e}\mathbb{E}[v_{\text{best}}]$ / $\frac{1}{e}$

Secretary Problem

恋爱不能靠瞎猜,要有科学的的37法则

2019年2月18日 记住这k个麦穗中的最大的麦穗,然后再继续前进,如果后面的麦穗有比这个还大的,那么摘取这个麦穗,这时这个麦穗是n个麦穗中的最大的概率为1/e,约为37%,简称37...

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- Simple $\frac{1}{e}$-rule.
- It has a wide culture influence.

Prophet Inequailty

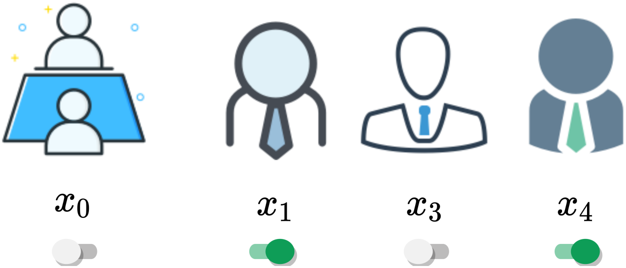
$v_0 \sim D_0$ $v_1 \sim D_1$ $v_2 \sim D_2$ $v_3 \sim D_3$

- Each candidate has a **independent** value $v_i \sim D_i$.
- They come in a **fixed** order.
- You must irrevocably decide which candidate to hire.
- Maximize $\mathbb{E}[v_{\text{hire}}]$.
- $\mathbb{E}[v_{\text{hire}}] \geq \frac{1}{2}\mathbb{E}[v_{\text{best}}]$.

Prophet Inequailty

- A simple strategy : Take the threshold T to be median of $D_{\max}$.
  - For each candidate i, it has at least $\frac{1}{2}$ probability to be looked at.
  - Then it is taken if it is larger than T.
  - $\frac{1}{2}T + \frac{1}{2}\sum_{i=1}^{n}\mathbb{E}[(v_i - T)^+] \geq \frac{1}{2}\mathbb{E}[(v^* - T)^+] + \frac{1}{2}T = \frac{1}{2}\mathbb{E}[v^*]$

(Online) Contention Resolution Schemes



- Each candidate is active **independently** (e.g. leaves you a good impression) w.p. x_i.
- $\sum_i x_i \leq 1$
- They come in a **fixed** order.
- You must irrevocably decide which candidate to hire.
- Maximize "selectability" $\min_i \Pr[i \text{ is hired} | i \text{ is active}]$.

A simple $\frac{1}{4}$-selectable OCRS

- Flip a coin for each candidate ($\frac{1}{2}$ head $\frac{1}{2}$ tail).
- Hire an active candidate only when its coin is head.
- Probability candidate i is looked at is at least $\frac{x_1+\ldots x_{i-1}}{2} \leq \frac{1}{2}$. Then it is hired when active w.p. $\frac{1}{2}$ conditioning on it is looked at.
- $\Pr[i \text{ is hired} | i \text{ is active}] \geq \frac{1}{4}$.

OCRS $\rightarrow$ Prophet inequality

- Idea: Resample v_i' from each D_i.
- Let i be active if v_i is the maximum in (v_i, v_{-i}').
- Use different samples for each i.
- Run online contention resolution schemes. [1]
- $O(n)$ samples are needed.

[1] There is a reduction in the reversed direction. Basically write OCRS as an LP and use prophet inequality as its seperation oracle.

Other applications of OCRS

- Rounding fractional solutions in discrete optimization.
  - View each variable as the proability that corresponding candidate is active.
  - Turn ex-ante feasibility into ex-post feasibility.

Matroid

- A matroid is $\mathcal{M} = (U, \mathcal{I})$ where $\mathcal{I} \subseteq 2^U$ is the set of independent sets.
  - Example: Trees in graph
- Its polytope $\mathcal{P}_{\mathcal{M}} = \{x \in [0,1]^n | \sum_{i \in S} x_i \leq \text{rank}(S), \forall S \subseteq U\}$.
  - Example: For all subset S of edges, if they have n distinct vertices, $x(S) \leq n - 1$.

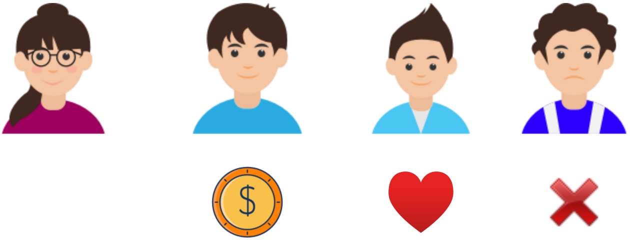
Matroid

- The problems above generalize to matroids.
  - Hire only one candidate $\rightarrow$ Hire an independent set of candidates
  - For (Online) CRS : $\sum_i x_i \leq 1 \rightarrow x \in \mathcal{P}_{\mathcal{M}}$
- Current status:
  - 2-competitive matroid prophet inequality exists.
  - 2-selectable matroid OCRS exists.
  - Constant-competitive matroid secretary is open for more than 20 years!

Sample Complexity

- In reality, knowing the full distribution information is hard.
- Instead, we take sample from each D_i.
  - A sample here is defined as n draws from all n distibutions $D_1, D_2, \ldots, D_n$ (or $x_1, \ldots, x_n$ if it is OCRS).
- There is a single-sample 2-competitive prophet inequality for single item!
  - Very simple algorithm : take the maximum of your sample as the threshold
- If the algorithm needs no sample at all, it is called **oblivious**.
  - Note the $\frac{1}{4}$-selectable OCRS is oblivious.
  - There is an $\frac{1}{2}$-selectable OCRS which selects each candidate with probability $\frac{1}{2-\sum_{j<i} x_j}$.
  - Is $\frac{1}{4}$ the best we can do when we have no information?

Optimal single-item oblivious OCRS

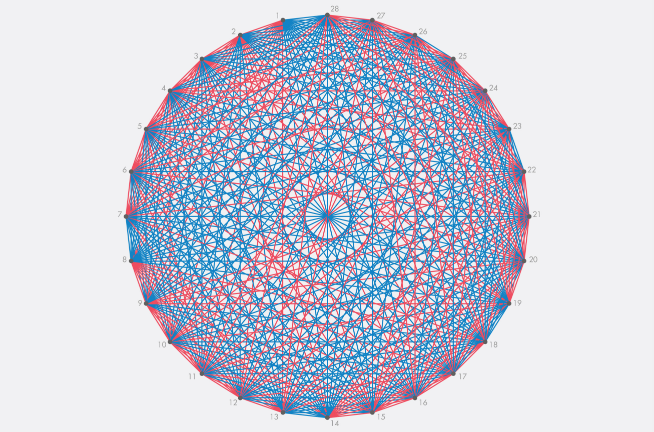


- Accept the first one w.p. $\frac{1}{2}$.
- Accept the second one w.p. 1 (if has rejected the first one).
- Has selectability $\Pr[i \text{ is hired} | i \text{ is active}] \geq \frac{1}{e}$ (by calculation)
- One can prove that any such "counting strategy" cannot do better than $\frac{1}{e}$ on uniform instance (by calculation).

Optimal single-item oblivious OCRS

- Is this the best we can do?
  - Intuition : The last one should be selected with probability 1.
  - Maybe there is a better strategy which utilize the index of candidates.
- It turns out "counting strategy" is the best we can do!

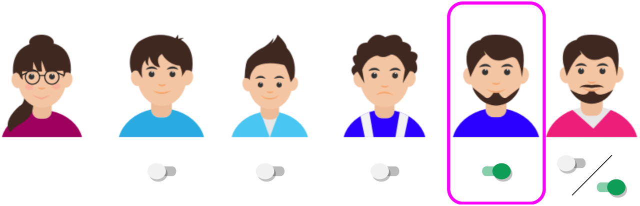
(Hypergraph) Ramsey theory



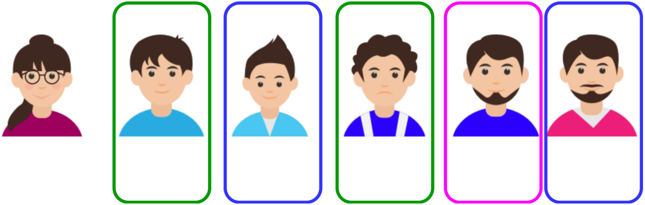
- If you color a sufficiently large complete (Hyper)graph with finitely many colors, there must be a monochromatic clique.

Simulate "counting strategy"

- Consider the probability of accept the first active candidate

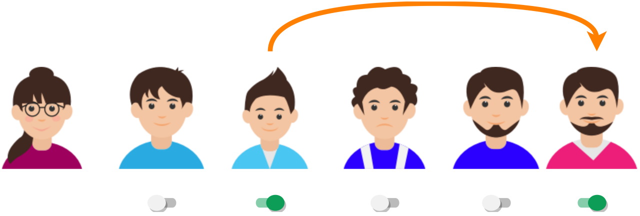


- Color each candidate according to $\lfloor \frac{p}{\epsilon} \rfloor$

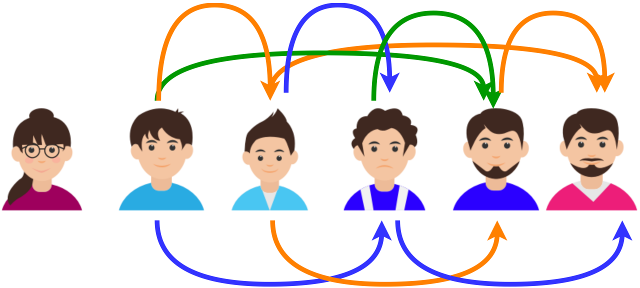


Simulate "counting strategy"

- What about the second candidate?



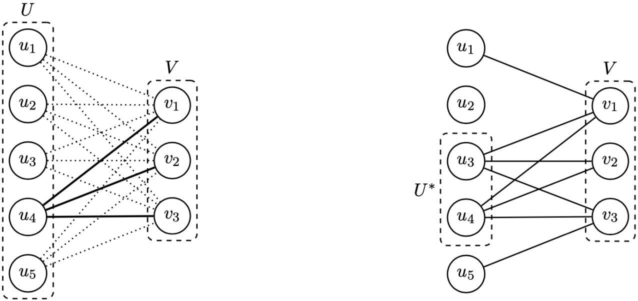
- Color the edge according to $\lfloor \frac{p}{\epsilon} \rfloor$



Simulate “counting strategy”

- More than two person : hypergraph
- Fill in the uniform instance in the monochromatic clique.

Oops! No oblivious OCRS for matroid



(a) The bipartite complete graph $K_{N,M}$. Here $i = 4$, edges adjacent to u_i has probability $x_e^i = 1$ of being active, while other edges each only has probability $1/M$ of being active. Here N should be a large enough number such that $N \gg M^M$.

(b) A realization $R(\boldsymbol{x})$ of this instance. U^* is the set of all vertices on left side of degree M. If N is large enough there will be many vertices happen to be in U^*. These vertices in U^* are indistinguishable to CRS, and u_4 ($i = 4$) is hidden between them.

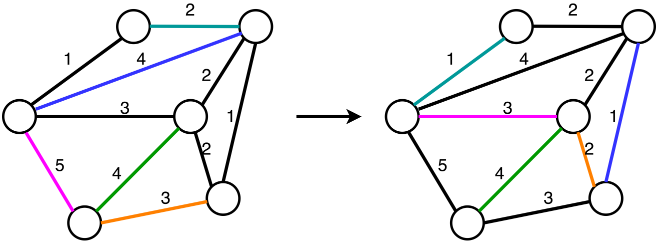
Figure 1: The hard instance for graphic matroids

Oops! No oblivious OCRS for matroid

- However, there is a matroid OCRS with $O(\log n)$ samples.
- It directly follow from the explicit construction of $\frac{1}{4}$-selectable matroid OCRS.

A matroid exchange Lemma

- For maximum weighted basis B and any basis B'. There is a bijection $f : B \to B'$ such that $B' - f(x) + x$ is a still a basis, and $w(f(x)) \leq w(x)$.



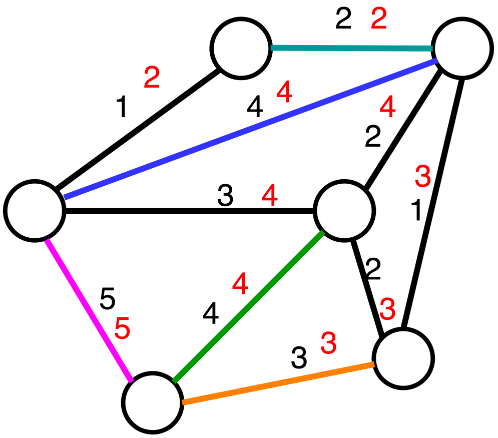
- The direction is important. (The reversed direction is trivial)

Prophet inequality with $O(\log n)$ samples

- Use resample as criterion for activation needs $O(n)$ samples.
- Idea: learn certain quantile as threshold for activation.
- Indepdence issue
- For simplicity, we describe our algorithm on graphs. It is the same on matroids.

Prophet inequality with $O(\log n)$ samples

- First, let the threshold of an edge be the median of its counterpart in optimal basis.



Prophet inequality with $O(\log n)$ samples

- We say an edge is active if it is larger than its threshold.
- Run OCRS with $O(\log n)$ samples.
- Indepdence is guarenteed since learning thresholds is seperated from the rest.

Prophet inequality with $O(\log n)$ samples

- Analysis
  - $x_i = \Pr[v_i \geq T_i] \leq 2\Pr[i \in \mathsf{OPT}]$. This is in the polytope (maybe after shrinking).
  - By selectability of OCRS, we get at least $\frac{1}{4}$ of

$$\sum_i \mathbb{E}[v_i I[v_i \geq T_i]]$$

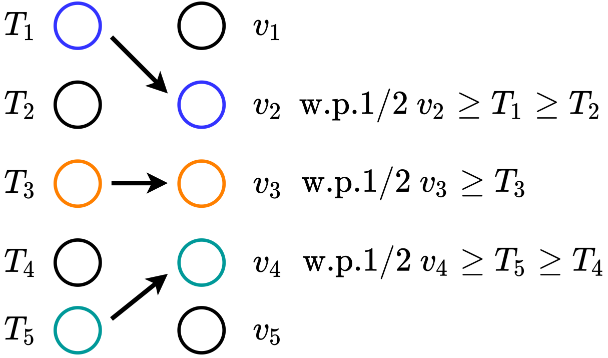
  - What about the rest? It would be a serious problem if

$$\sum_{i \in \mathsf{OPT}} \mathbb{E}[v_i I[v_i < T_i]]$$

    contributes a lot to OPT.

Prophet inequality with $O(\log n)$ samples

- Analysis
  - Luckily, it cannot be the case! We prove this by matching with the maximum weighted basis w.r.t. weight T_i.



Open Problems

- Matroid secretary
- Is there matroid prophet inequality from constant samples?
- Can we explore constrained order version of OCRS?

Q & A

Questions?

Thank you!