Near-optimal Algorithm for Constructing Greedy Consensus Tree

Hongxun Wu

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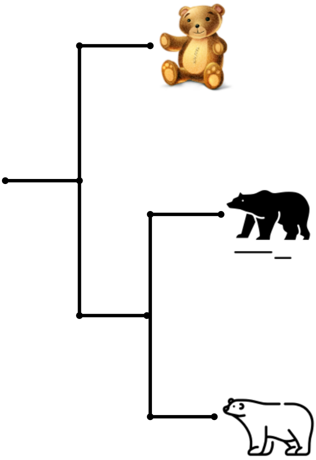
- **Leaves** of the tree represent species.
- Each **inner node** represents the least common ancestor of all leaves in its subtree.



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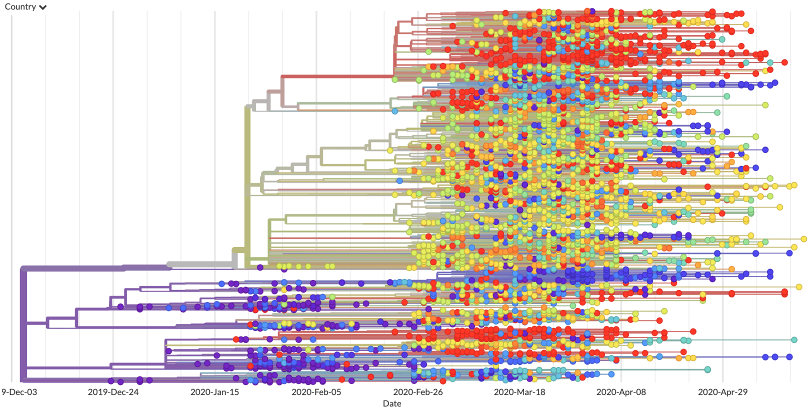

Figure: Phylogenetic Tree of Covid-19[1]

[1] Genomic epidemiology of novel coronavirus. 2020. URL: https://nextstrain.org/ncov/global.

Consensus tree

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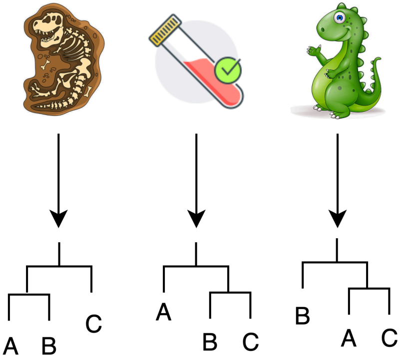

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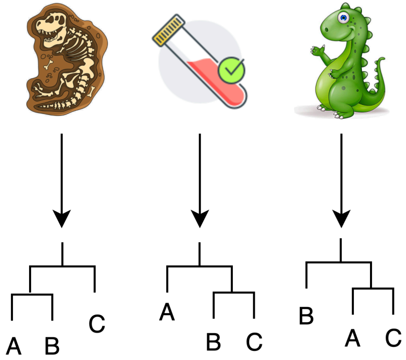

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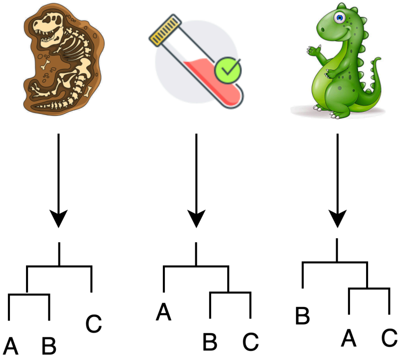

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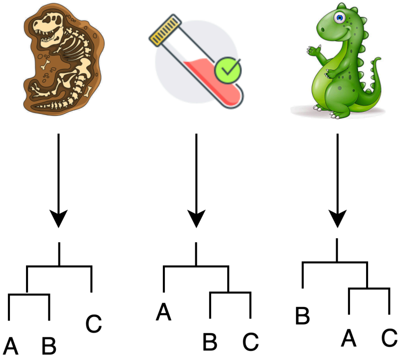
- The input size is $\Theta(kn)$.



Consensus tree

- Many consensus tree methods were proposed.

Consensus tree method	Running time
Adam's consensus tree	$O(kn \log n)$
Strict consensus tree	$O(kn)$
Loose consensus tree	$O(kn)$
Frequency difference consensus tree	$O(kn \log^2 n)$
Majority-rule consensus tree	$O(kn \log k)$, Randomized $O(kn)$
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- Most of them have near-optimal running time. Greedy consensus tree is one of the exceptions.

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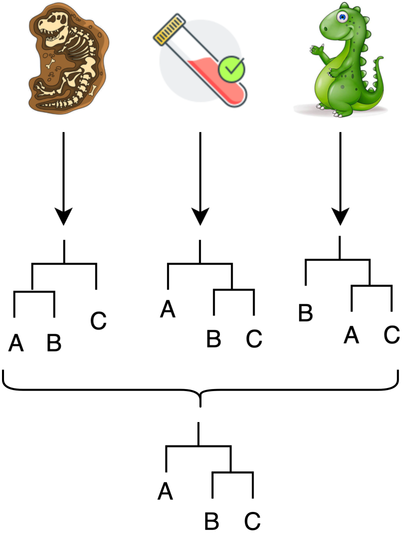

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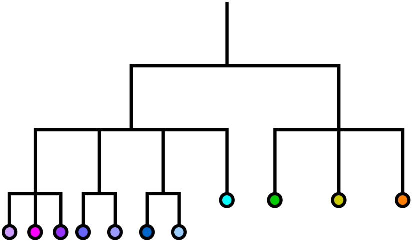

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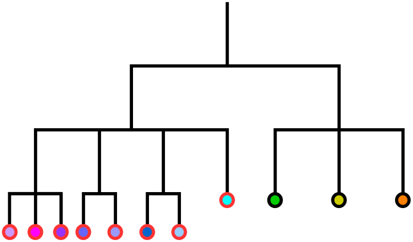

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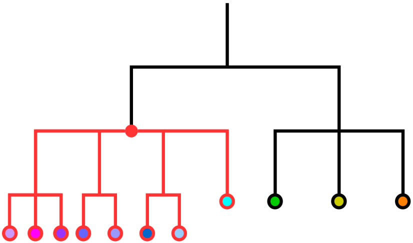

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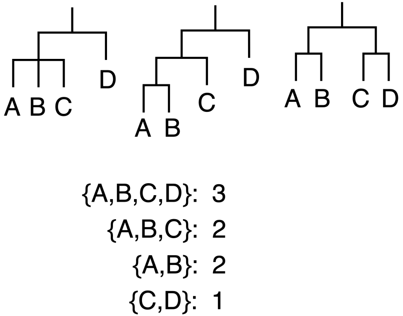

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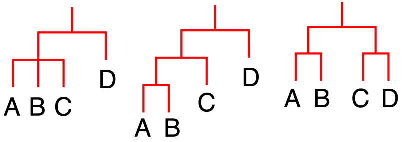

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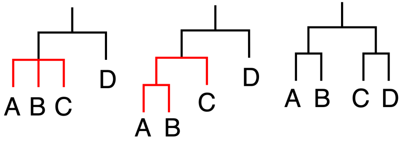

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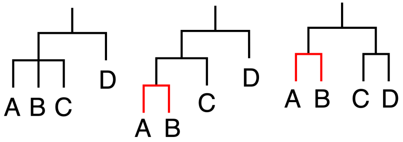

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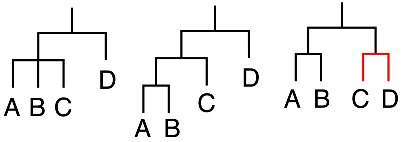

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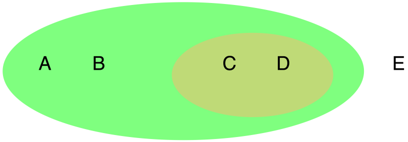

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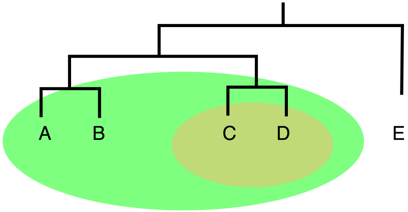

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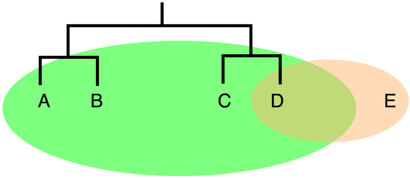

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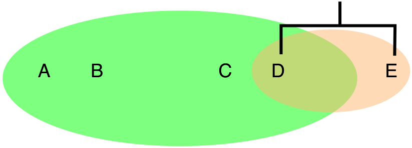

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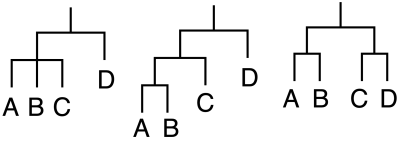

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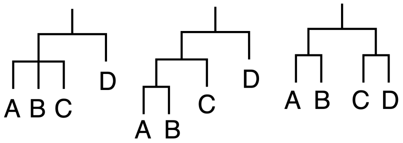

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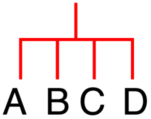

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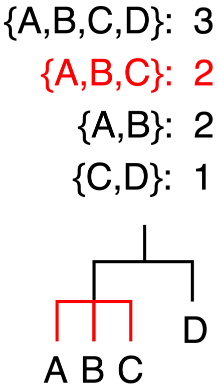

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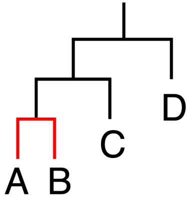

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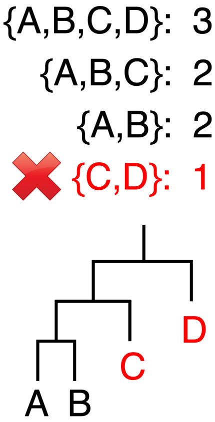
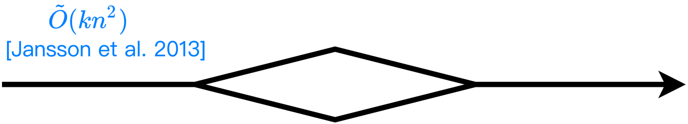


Previous Works



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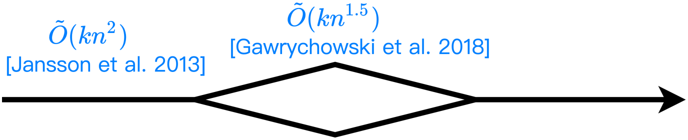

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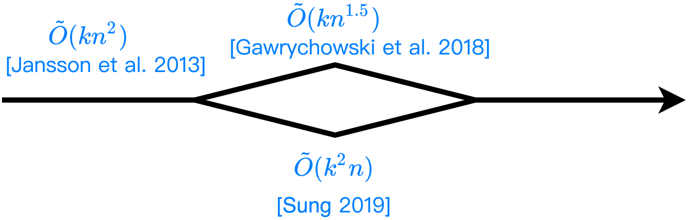

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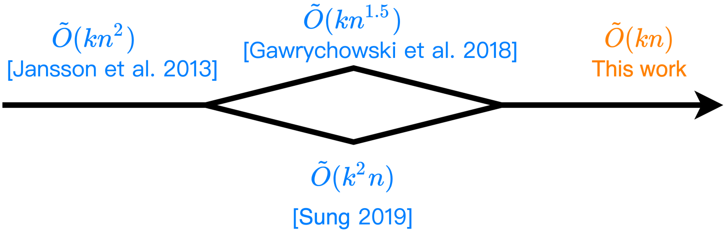

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- Handle first part within $\tilde{O}(kn)$ time is simple.
- The hard part is to determine whether each cluster is consistent with our current consensus tree.

Characterization of consistency

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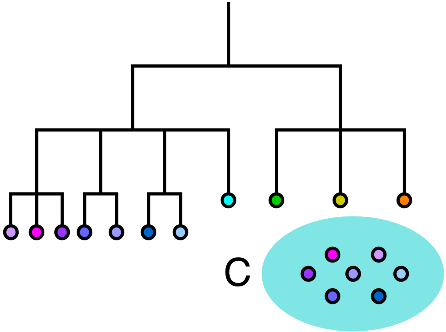
Let v be the LCA of cluster C on consensus tree



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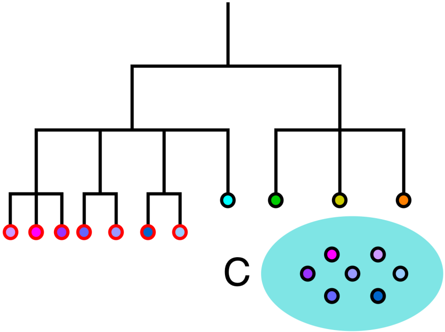

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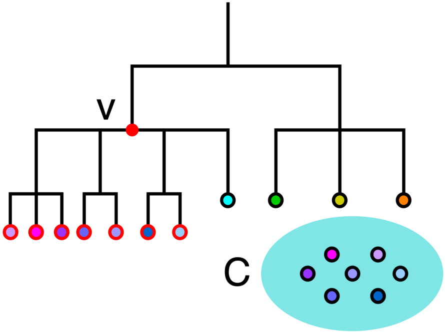

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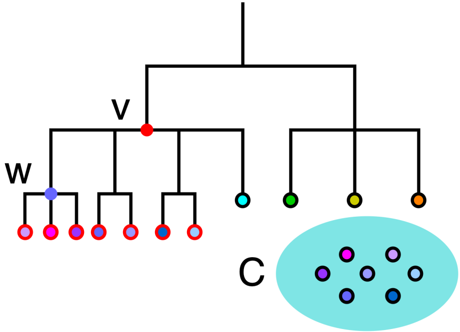

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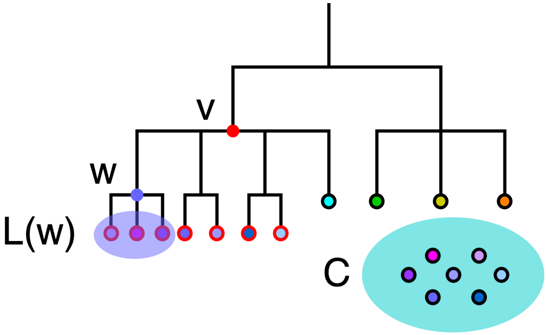

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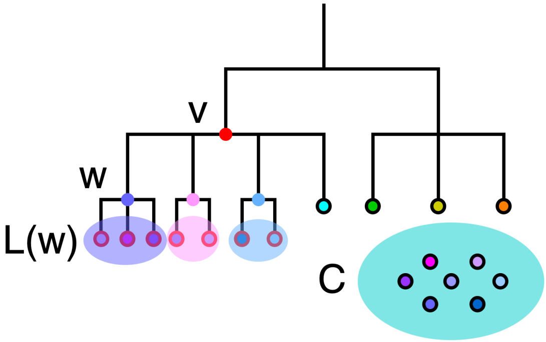
C is consistent with the consensus tree if and only if it is the union of several such $L(w)$.



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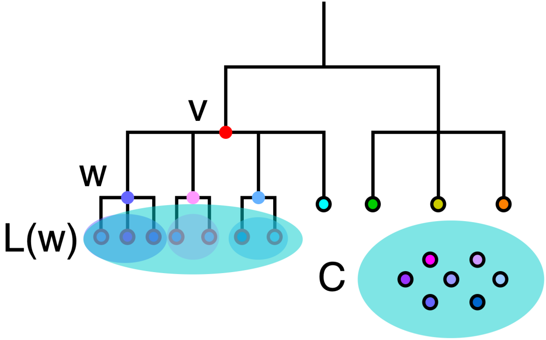

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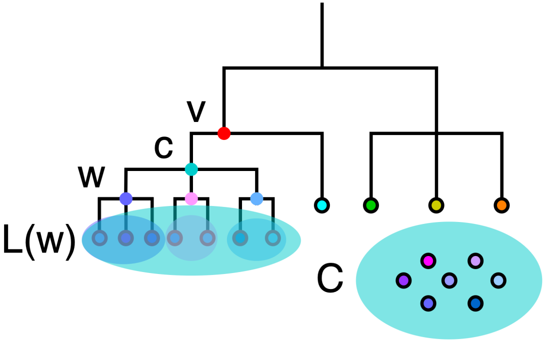

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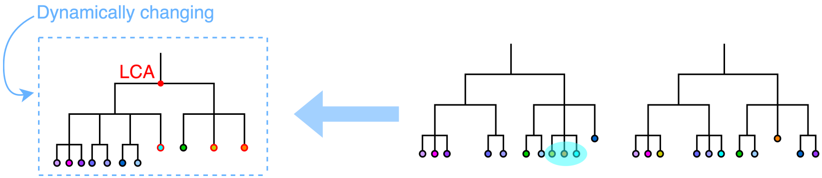

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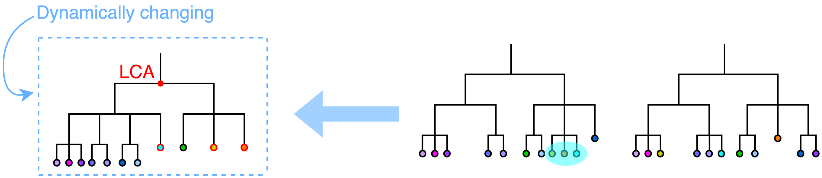

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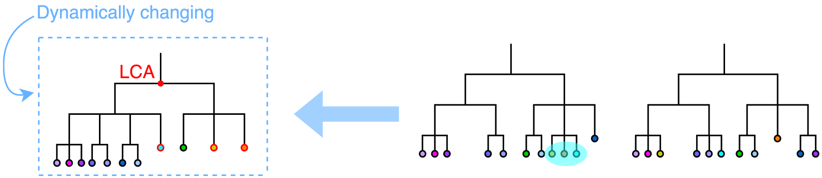

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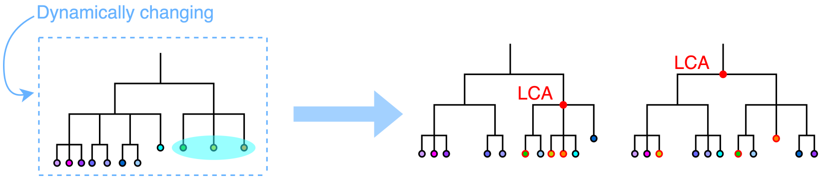

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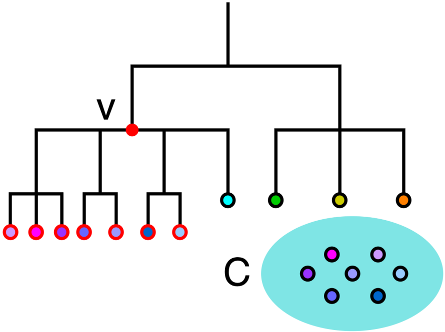

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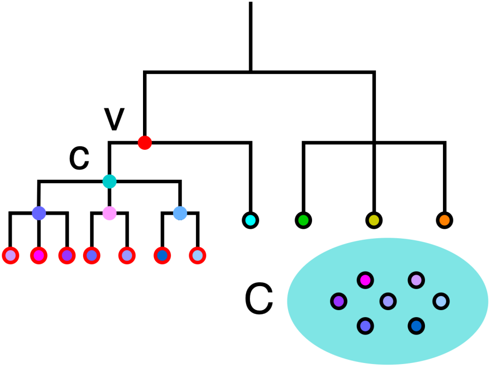

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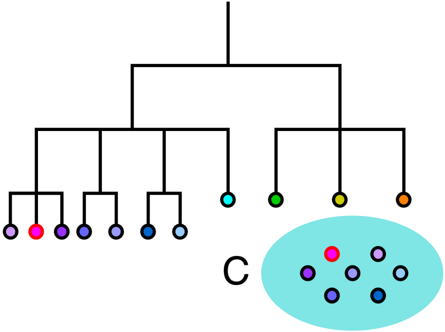

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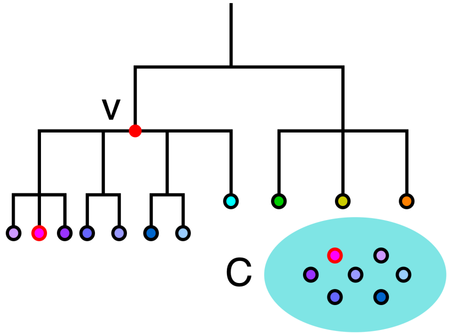

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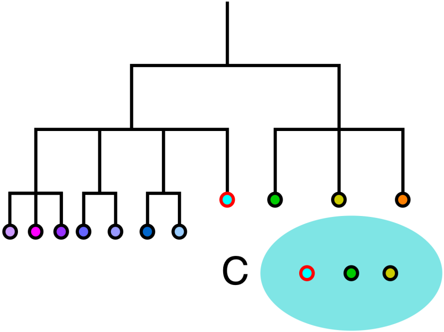

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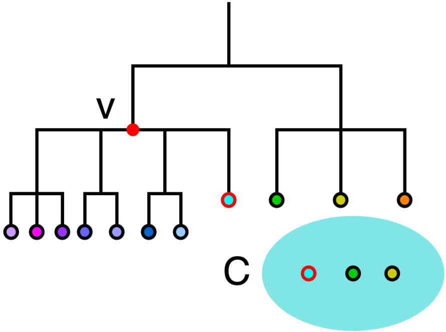

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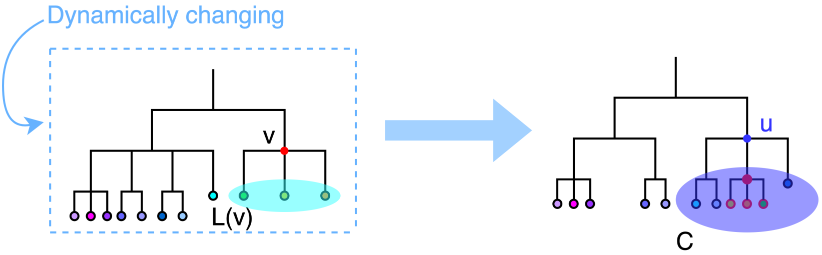

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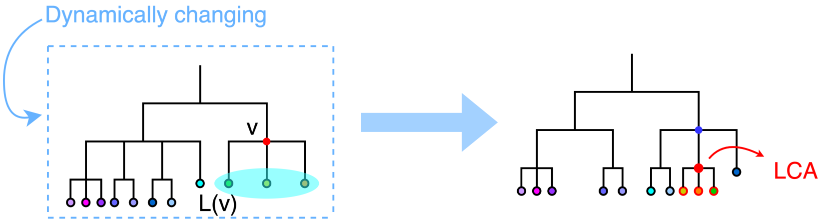

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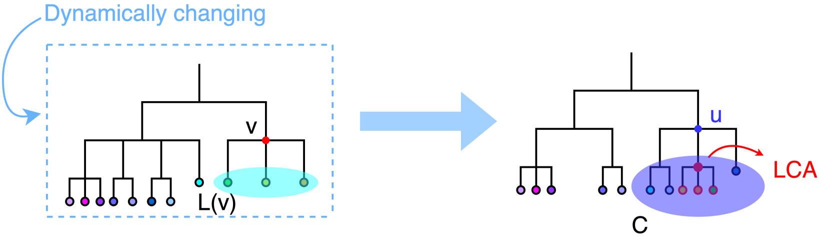

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- As a result, the modified characterization can be implemented by maintaining LCAs of clusters in the dynamic tree on k static trees.

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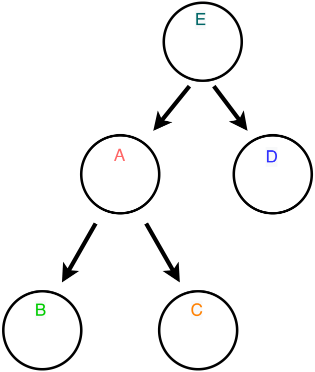

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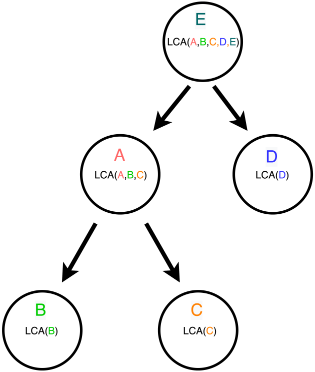

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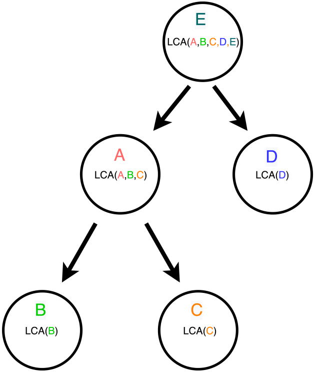

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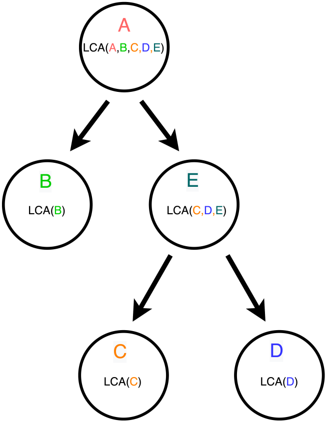
- When the children of a node changes, we recompute the LCA of its two children.



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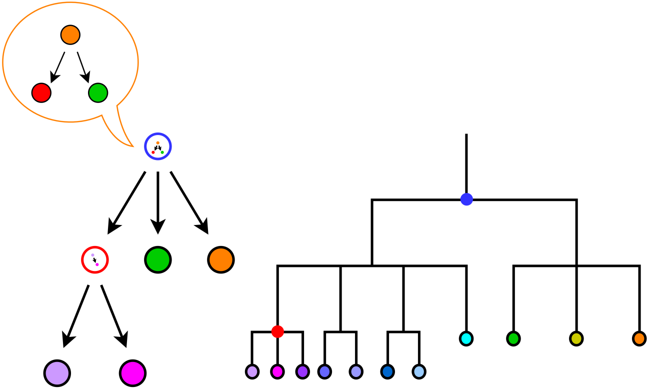

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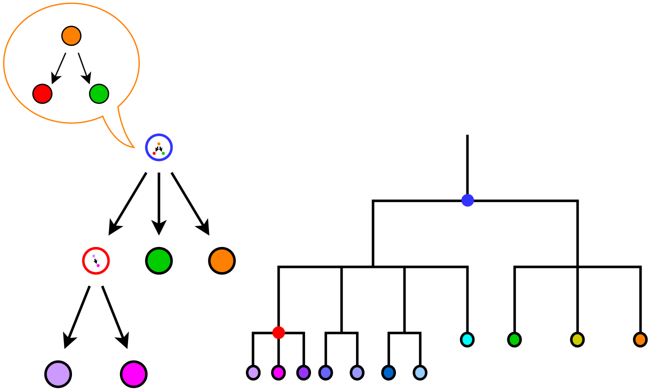
  - Each node of the BST is the LCA of one of its children.



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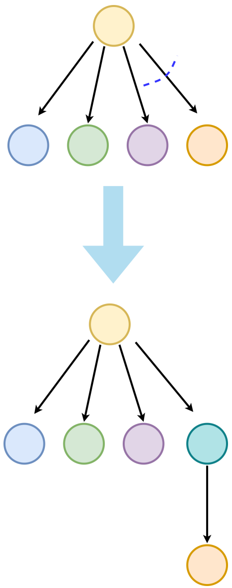

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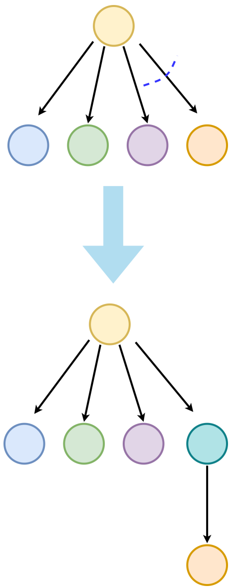

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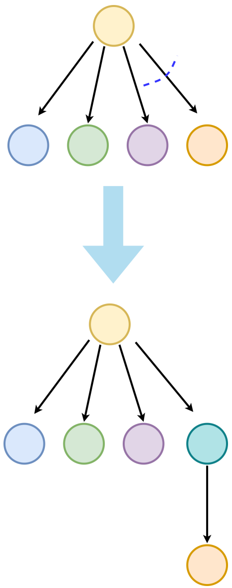

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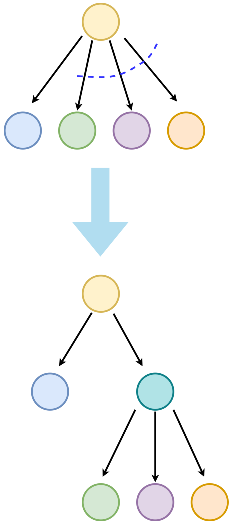

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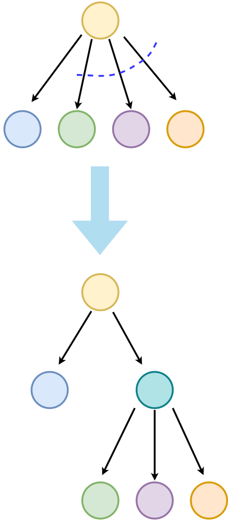

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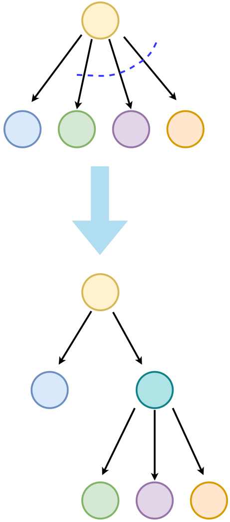
- Since we always enumerate the smaller part, in total $O(kn \log n)$ deletions in BSTs.



Q & A

Questions?

Thank you!